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$$B_j(m, n; \nu, g_a)/K_j \ll \exp\left\{-c_5 \frac{(\log K_j)^3}{(\log x_2)^2}\right\} \ll 1/(\log x)^4$$

provided $c(\alpha)$ is large enough. By (62), we infer that we have uniformly for $1 \leqslant \nu \leqslant x$, $0 \leqslant y \leqslant 4$,

$$H_\nu(x, y) \ll (\log x)^{2y/3} (\log_2 x)^{y/4} . \tag{73}$$

Inserting the above estimate into (24) with, say, $T := \log x$, we see that we may choose

$$E_2(x, y) \asymp (\log x)^{11y/12 - 1} (\log_2 x)^{2 + y/4} , \tag{74}$$

which is hence slowly increasing. Therefore we get by Theorem 7 that

$$\Delta(n, g_\alpha) < \tau(n) (\log n)^{11y/24 - 1/2 - (1/2)\log y + o(1)} \qquad \text{pp}l . \tag{75}$$

The required estimate (59) now follows on taking optimally $y = \frac{12}{11}$.

4. FUNCTIONS OF EXCESSIVE GROWTH: THE CASE $f(d) = \theta d$

We now investigate, in a quantitative form, the uniform distribution on divisors of the function

$$h_\theta(d) := \theta d$$

when θ is a given irrational real number. This study is similar in principle to that of the previous section, but more complicated inasmuch as the effective bounds for $\Delta(n; h_\theta)$ will depend on the arithmetic nature of θ. On the other hand we shall not need, as might be expected, any involved tool for the estimation of the relevant exponential sums.

More explicitly, let us define $Q(x) := x/(\log x)^{10}$, and

$$q(x; \theta) := \inf\{q : 1 \leqslant q \leqslant Q(x),\ \| q\theta \| \leqslant 1/Q(x)\} \tag{76}$$

where $\| u \|$ denotes the distance of u to the set of integers. Our results depend on a free parameter $y, 0 < y \leqslant 4$, and may be expressed conveniently in terms of any increasing lower bound for $q(x; \theta)$, say $q^*(x; y, \theta)$, with the property that $q^*(x; y, \theta)/(\log x)^{y/4}$ is decreasing. A possible choice is

$$q^*(x; y, \theta) := 4(\log x)^{y/4} \Big/ \int_1^x \frac{y(\log t)^{y/4 - 1}}{t \inf_{u \geqslant t} q(u; \theta)} \, \mathrm{d}t . \tag{77}$$

Unless θ has abnormally good rational approximations, we have

$$q^*(x; y, \theta) \asymp (\log x)^{y/4} . \tag{78}$$

Indeed, let us define, for real positive γ, the set

$$E(\gamma) := \{\theta \in \mathbf{R}\setminus\mathbf{Q} \colon \liminf_{x\to\infty} q(x;\theta)/(\log x)^{\gamma} > 0\} .$$

Then $E(\gamma)$ contains almost all real numbers, and in particular, by Liouville's theorem, all algebraic numbers. Moreover, it is not difficult to show that $\mathbf{R}\setminus E(\gamma)$ has zero Hausdorff dimension. We readily see from (77) that (78) holds for all $\theta \in E(\gamma)$ whenever $\gamma > \frac{1}{4}y$.

We shall establish the following result.

THEOREM 12. *Let* $\theta \in \mathbf{R}\setminus\mathbf{Q}$, $0 < \delta < 1$. *Uniformly for* $x \geqslant 2$, $0 < y \leqslant 4$, *we have*

$$\sum_{n\leqslant x} \left(\frac{y}{4}\right)^{\Omega(n)} \frac{\Delta(n;h_\theta)^2}{n} \ll (\log x)^y/q^*(x;y,\theta)^{\delta} . \tag{79}$$

Taking $y = 1$, we immediately obtain an effective uniform distribution result which is valid without any restriction on θ. The corresponding qualitative result had been established by Dupain, Hall & Tenenbaum [4].

COROLLARY 7. *The function* $h_\theta(d) = \theta d$ *is* erd *for each irrational number* θ. *Moreover, if* $0 < \delta < 1$, *then we have*

$$\Delta(n;h_\theta) < \tau(n)/q^*(n;1,\theta)^{\delta/2} \quad \mathrm{pp}l .$$

The above bound is always $o(\tau(n))$ and $\ll \tau(n)/(\log n)^{-\delta/8}$ for $\theta \in E\left(\frac{1}{4}\right)$. However, if we are prepared to exclude a set of θ of Hausdorff dimension zero, we may achieve a better pp$l$ estimate by taking $y = \frac{4}{3}$ in (79). Indeed, the following statement stems from Theorem 12 by optimising the parameter y under the assumption that (78) holds.

COROLLARY 8. *Let* $\gamma > \frac{1}{3}$. *Then we have for all* $\theta \in E(\gamma)$

$$\Delta(n;h_\theta) < \tau(n)^{(\log 3)/\log 4 + o(1)} \quad \mathrm{pp}l .$$

It is very likely that the estimate $\Delta(n,h_\theta) < \tau(n)^{1/2+o(1)}ppl$ holds outside a set of θ with Hausdorff dimension 0, but this is beyond the scope of the method employed here. If we only require that the set of exceptional θ have Lebesgue measure zero, this last bound does actually hold and can be easily established by the variance argument used for the proof of Theorem 14 below. Moreover, with this level of generality, the exponent $1/2$ is sharp.

Of course, Corollaries 7 and 8 may be used to exhibit Behrend sequences. An immediate application of these results and Theorem 3 yields the following proposition.

COROLLARY 9. *Let* $\theta \in \mathbf{R}\backslash\mathbf{Q}$, $0 < \delta < 1$. *Then the sequence*

$$\mathscr{U}(\theta, \delta) := \{n \geqslant 2 : \langle \theta n \rangle \leqslant q^*(n; 1, \theta)^{-\delta/2}\}$$

is a Behrend sequence. Furthermore, if $\theta \in E(\gamma)$ *for some* $\gamma > \frac{1}{3}$, *and in particular if* θ *is algebraic, then the sequence*

$$\mathscr{W}(\theta, \rho) := \{n \geqslant 2 : \langle \theta n \rangle \leqslant (\log n)^{-\rho}\}$$

is a Behrend sequence for all $\rho < \frac{1}{2} \log \frac{4}{3}$.

Let $\rho^*(\gamma)$ denote the supremum of those exponents ρ such that $\mathscr{W}(\theta, \rho)$ is Behrend for all $\theta \in E(\gamma)$. The above result implies that $\rho^*(\gamma) \geqslant \frac{1}{2} \log \frac{4}{3} \approx 0 \cdot 14384$ for $\gamma > \frac{1}{3}$, and it is natural to conjecture that there exists a γ_0 such that $\rho^*(\gamma) = \log 2$ for $\gamma > \gamma_0$. By techniques similar to those presented below for the proof of Theorem 12, it can be shown that the distributions of $\Omega(n)$ and $\langle \theta n \rangle$ are largely independent. Moreover, using Vaughan's bound for exponential sums over primes (see e.g. Davenport [1], chapter 25), this statement can be put in an effective form which is sufficiently strong to yield

$$\sum_{n \in \mathscr{W}(\theta, \rho)} \frac{(\log n)^{\log 2 - 1/2}}{n 2^{\Omega(n)}} < \infty$$

for all $\rho > \log 2$ and, e.g., $\theta \in E(2)$. By a result of Hall ([12], theorem 1), this implies that, when $\theta \in E(2)$ and $\rho > \log 2$, the sequence $\mathscr{W}(\theta, \rho)$ is not Behrend. A weak consequence of this is that $\rho^*(\gamma) \leqslant \log 2$ for all $\gamma \geqslant 2$.

For the proof of Theorem 12, we have chosen to avoid some technical complications by applying Theorem 8 rather than Theorem 7, although the latter could in principle lead to better quantitative estimates. In connection with the general upper bound (28) for weighted logarithmic averages of $\Delta(n; f)^2$, we introduce the expressions

$$(80) \quad T(x; z, \theta) := \sum_{n \leqslant x} z^{\Omega(n)} \frac{e(\theta n)}{n}, \; S(x, z, \theta) := \sum_{k \leqslant x} \frac{z^{\Omega(k)}}{k} \, | \, T(x/k; z, k\theta) \, |,$$

so that (28) reads, for $f = h_\theta$,

$$(81) \quad \sum_{n \leqslant x} \left(\frac{y}{4}\right)^{\Omega(n)} \frac{\Delta(n; h_\theta)^2}{n} \ll (\log x)^y \left\{ \frac{1}{T^2} + \log T \sum_{1 \leqslant \nu \leqslant T} \frac{1}{\nu} \, \varepsilon_\nu^+(x, y; h_\theta) \right\},$$

uniformly for $x \geqslant 2$, $0 \leqslant y \leqslant y_0 < 8$, $T \geqslant 2$, where $\varepsilon_\nu^+(x, y; h_\theta)$ is any non-increasing function of x such that $x \mapsto \varepsilon_\nu^+(x, y; h_\theta)(\log x)^{y/2}$ is non-decreasing and satisfies

$$S\left(x; \tfrac{1}{4}y, \nu\theta\right) \ll (\log x)^{y/2} \varepsilon_\nu^+(x, y; h_\theta) .$$

For technical reasons, it will be more convenient to use at certain stages Cesàro-type averages, so we set

$$T^*(x; z, \theta) := \sum_{n \leqslant x} z^{\Omega(n)} e(\theta n), \tag{82}$$
$$S^*(x; z, \theta) := \sum_{k \leqslant x} z^{\Omega(k)} \left| T^*(x/k; z, k\theta) \right| ,$$

from which we shall derive information on the quantities in (80) by partial summation.

We need several preliminary estimates which we state as independent lemmata.

LEMMA 3. *For* $0 \leqslant z \leqslant 1$, $1 \leqslant a \leqslant q \leqslant x$, $(a, q) = 1$, $|\theta - a/q| \leqslant 1/q^2$, *we have*

$$S^*(x; z, \theta) \ll x(\log x)^{z-1}(\log_2 x)^z + x(\log x)^2 \left\{ \sqrt{\frac{q}{x} + \frac{1}{\sqrt{q}}} \right\} . \tag{83}$$

Proof. This is a variant of a familiar lemma in Vinogradov's method. We first note the trivial estimate.

$$\left| T^*(w; z, \theta) \right| \leqslant \sum_{n \leqslant w} z^{\Omega(n)} \ll w(\log w)^{z-1} \qquad (w \geqslant 2) , \tag{84}$$

which stems from (6) or e.g. theorem III.3.5 of [25]. Then, assuming, as we may, that x is large, we put $y = (\log x)^6 \leqslant \sqrt{x}$ and we split the outer k-sum in $S^*(x; z, \theta)$, applying (84) with $w = x/k$ for the ranges $k \leqslant y$ and $x/y < k \leqslant x$. Using (84) again with partial summation for the corresponding resulting summation over k, and bounding $z^{\Omega(k)}$ by 1 in the complementary sum, we arrive at

$$S^*(x; z, \theta) \ll x(\log x)^{z-1}(\log_2 x)^z + \sum_{0 \leqslant j < J} W(2^j y) , \tag{85}$$

with $J := \dfrac{\log (x/y)}{\log y}$ and

$$W(K) := \sum_{K < k \leqslant 2K} \left| T^*(x/k; z, k\theta) \right| . \tag{86}$$

Now we have for $y \leqslant K \leqslant x/y$, by the Cauchy-Schwarz inequality,

$$W(K)^2 \leqslant K \sum_{K < k \leqslant 2K} \sum_{1 \leqslant m, n \leqslant x/k} z^{\Omega(mn)} e(k\theta(n-m))$$
$$= K \sum_{1 \leqslant m, n \leqslant x/K} z^{\Omega(mn)} \sum_{K < k \leqslant \min(2K, x/m, x/n)} e(k\theta(n-m))$$
$$\ll K \sum_{1 \leqslant m, n \leqslant x/K} \min(K, 1/\|\theta(n-m)\|) \tag{87}$$

To estimate the h-sum, we write $\theta = a/q + \beta$ with $|\beta| \leqslant 1/q^2$ and $h = tq + r$ with $0 \leqslant r < q$. Then, for each given t, we have $\|\theta h\| = \|\alpha_r\|$ with $\alpha_r := ra/q + r\beta + tq\beta$. For $0 \leqslant r \neq s < q$, and if $\langle\alpha_r\rangle - \frac{1}{2}$ and $\langle\alpha_s\rangle - \frac{1}{2}$ have the same sign, we may write $\big|\|\alpha_s\| - \|\alpha_r\|\big| = \|\alpha_s - \alpha_r\| > \|(s-r)a/q\| - 1/q$. Hence there are at most 6 values of $r, 0 \leqslant r < q$, such that α_r belongs to any given interval $(v/q, (v+1)/q]$ modulo 1. This implies that

$$\begin{aligned}\sum_{0 \leqslant h \leqslant x/K} \min(K, 1/\|\theta h\|) &\ll \sum_{0 \leqslant t \leqslant x/Kq} \sum_{0 \leqslant r < q} \min(K, 1/\|ar/q\|) \\ &\ll (1 + x/Kq)(K + q\log q) \\ &\ll x\log x\left(\frac{K}{x} + \frac{q}{x} + \frac{1}{q} + \frac{1}{K}\right) \\ &\ll x\log x\left(\frac{1}{y} + \frac{q}{x} + \frac{1}{q}\right).\end{aligned}$$

By (87), we infer that

$$W(K) \ll x\log x\left\{\sqrt{\frac{q}{x}} + \frac{1}{\sqrt{q}}\right\} + \frac{x}{(\log x)^2} \quad (y \leqslant K \leqslant x/y). \tag{88}$$

Inserting this into (85), we readily get the required estimate.

LEMMA 4. *Let* $\theta \in \mathbf{R}\setminus\mathbf{Q}$. *For all* $x \geqslant 3$ *such that* $q(x;\theta) > (\log x)^{10}$ *and uniformly for* $0 \leqslant z \leqslant 1$, $1 \leqslant \nu \leqslant \log x$, *we have*

$$S^*(x; z, \nu\theta) \ll x(\log x)^{z-1}(\log_2 x)^z. \tag{89}$$

Proof. Let $q = q(x;\theta)$. Then, by Dirichlet's theorem, $q \leqslant Q(x) = x/(\log x)^{10}$ and, for suitable integer a, we have $|\theta - a/q| \leqslant 1/qQ \leqslant 1/q^2$. Moreover the minimality assumption on q implies that $(a, q) = 1$. Thus for each ν with $1 \leqslant \nu \leqslant \log x$ we have $|\nu\theta - a_\nu/q_\nu| \leqslant 1/q_\nu^2$ for some integers a_ν, q_ν with $(a_\nu, q_\nu) = 1$, $(\log x)^9 < q_\nu \leqslant q$. Applying Lemma 3, we obtain

$$S^*(x; z, \nu\theta) \ll x(\log x)^{z-1}(\log_2 x)^z + x/(\log x)^{5/2}.$$

The result follows.

Our next lemma concerns the distribution of the numbers $z^{\Omega(n)}$ on arithmetic progressions. We put

$$H(x; z; q, a) := \sum_{\substack{n \leqslant x \\ n \equiv a \pmod q}} z^{\Omega(n)}, \qquad H(x; z; q) := \sum_{\substack{n \leqslant x \\ (n,q)=1}} z^{\Omega(n)}. \tag{90}$$

LEMMA 5. *Let* $A > 0$. *Then there is a positive constant* c *such that, for* $0 \leqslant z \leqslant 1$, $x \geqslant 2$, $1 \leqslant q \leqslant (\log x)^A$, $(a, q) = 1$, *we have*

$$(91) \qquad H(x; z; q, a) = \frac{1}{\varphi(q)} H(x; z; q) + O(x \exp\{-c(\log x)^{1/3}\}) .$$

Proof. It would be possible, as in Rieger [21], to obtain an exponent $\frac{1}{2}$ instead of $\frac{1}{3}$ in the remainder term using contour integration and standard analytic information on powers $L(s, \chi)^z$ of Dirichlet L-functions. The result stated will be more than sufficient for our actual purpose. It may be given a short proof which we include for the convenience of the reader. We introduce the Dirichlet characters to the modulus q and write

$$(92) \qquad H(x; z; q, a) = \frac{1}{\varphi(q)} H(x; z; q) + O\left(\max_{\chi \neq \chi_0} \left| \sum_{n \leqslant x} \chi(n) z^{\Omega(n)} \right| \right),$$

where the maximum is taken over all non-principal characters χ modulo q. This remainder may be bounded above by appealing to the prime number theorem for arithmetic progressions in the form

$$(93) \qquad \sum_{p \leqslant t} \chi(p) \ll t e^{-c(b)\sqrt{\log t}} \qquad (q \leqslant (\log t)^b) ,$$

valid for any non-principal character χ to the modulus q. Here b is any fixed parameter and $c(b) > 0$. This estimate may be found e.g. in Davenport [1], p. 132.

We also introduce the largest prime factor function $P^+(n)$ and recall from [25] (theorem III.5.1) the estimate

$$(94) \qquad \Psi(x, y) := \sum_{\substack{n \leqslant x \\ P^+(n) \leqslant y}} 1 \ll x^{1 - 1/(2 \log y)} \qquad (x \geqslant 2, y \geqslant 2) .$$

For any non-principal character χ to the modulus q, we have

$$\begin{aligned}
\sum_{n \leqslant x} \chi(n) z^{\Omega(n)} &= \sum_{\substack{n \leqslant x \\ P^+(n) > \exp\sqrt{\log x}}} \chi(n) z^{\Omega(n)} + O(x e^{-\frac{1}{2}\sqrt{\log x}}) \\
&= \sum_{\substack{mr \leqslant x \\ P^+(mr) = r \\ r > \exp\sqrt{\log x}}} \chi(m)\chi(r) z^{\Omega(m)+1} + O(x e^{-\frac{1}{2}\sqrt{\log x}}) \\
&= \sum_{mP^+(m) \leqslant x} \chi(m) z^{\Omega(m)+1} \sum_{\substack{P^+(m) \leqslant r \leqslant x/m \\ r \text{ prime} \\ r > \exp\sqrt{\log x}}} \chi(r) + O(x e^{-\frac{1}{2}\sqrt{\log x}}) \\
&\ll \sum_{mP^+(m) \leqslant x} \frac{x}{m} e^{-c(2A)\sqrt{\log(x/m)}} + O(x e^{-\frac{1}{2}\sqrt{\log x}})
\end{aligned}$$

by (93). Write $c(2A) = c_1$ for brevity and set $M_j := x/e^j (j = 0, 1, \ldots)$. The above m-sum does not exceed

$$\sum_{0 \leqslant j \leqslant \log x} \frac{x}{M_j} e^{-c_1\sqrt{j}} \sum_{\substack{M_{j+1} < m \leqslant M_j \\ P^+(m) \leqslant e^{j+1}}} 1 \ll \sum_{0 \leqslant j \leqslant \log x} x e^{-c_1\sqrt{j} - (\log x)/(2j)},$$

by (94). Since $c_1\sqrt{j} + (\log x)/(2j) \gg (\log x)^{1/3}$, we obtain that the estimate

$$\sum_{n \leqslant x} \chi(n) z^{\Omega(n)} \ll x \exp\{-c(\log x)^{1/3}\}$$

holds, under the prescribed conditions, for a suitable positive constant c. In view of (92) this readily yields the required result.

LEMMA 6. *Let* $A > 0$. *There exists a positive constant* c_0 *such that, uniformly for* $0 \leqslant z \leqslant 1$, $x \geqslant 2$, $1 \leqslant q \leqslant (\log x)^A$, $(a, q) = 1$, *we have*

$$T^*(x; z, a/q) = \sum_{t \mid q} \frac{\mu(t) z^{\Omega(q/t)}}{\varphi(t)} H(tx/q; z; t) + O(x \exp\{-c_0(\log x)^{1/3}\}) .$$

Proof. We have

$$\begin{aligned} T^*(x; z, a/q) &= \sum_{0 \leqslant b < q} e(ab/q) \sum_{\substack{n \leqslant x \\ n \equiv b \,(\mathrm{mod}\, q)}} z^{\Omega(n)} \\ &= \sum_{0 \leqslant b < q} e(ab/q) z^{\Omega((b,q))} \sum_{\substack{m \leqslant x/(b,q) \\ m \equiv b/(b,q) \,(\mathrm{mod}\, q/(b,q))}} z^{\Omega(m)} \\ &= \sum_{t \mid q} \sum_{\substack{0 \leqslant h < t \\ (h,t) = 1}} e(ah/t) z^{\Omega(q/t)} H(tx/q; z; t, h) , \end{aligned}$$

where we have put $t = q/(b, q)$, $b = hq/t = h(b, q)$ with $(h, t) = 1$. Using Lemma 5 to evaluate $H(tx/q; z; t, h)$, and appealing to Ramanujan's formula

$$\sum_{\substack{0 \leqslant h < t \\ (h,t) = 1}} e(ah/t) = \mu(t) ,$$

we get

$$T^*(x; z, a/q) = \sum_{t \mid q} \frac{\mu(t) z^{\Omega(q/t)}}{\varphi(t)} H(tx/q; z; t) + O(qx \exp\{-c(\log x)^{1/3}\}) .$$

Hence the required estimate holds for all $c_0 < c$.

We are now in a position to evaluate $S^*(x; z, \nu\theta)$ when $q(x; \theta)$ is small.

LEMMA 7. *Let* $\theta \in \mathbf{R}\setminus\mathbf{Q}$, *and* $0 < \delta < 1$. *For all real numbers* $x \geqslant 3$ *such that* $q(x;\theta) \leqslant (\log x)^{10}$ *and uniformly for* $0 \leqslant z \leqslant 1$, $1 \leqslant \nu \leqslant \log 2q(x;\theta)$, *we have*

$$S^*(x;z,\nu\theta) \ll x(\log x)^{2z-1}\{(\log x)^{-\delta z} + q(x;\theta)^{-\delta}\}\,. \tag{95}$$

Proof. As previously, we start with Dirichlet's theorem which implies that $q = q(x,\theta) \leqslant Q(x) = x/(\log x)^{10}$. Hence we have $|\theta - a/q| \leqslant 1/qQ(x)$ with $(a,q) = 1$. For $1 \leqslant \nu \leqslant \log 2q$, we write $a_\nu := a\nu/(q,\nu), q_\nu := q/(q,\nu)$. Putting $Q := x/(\log x)^{11}$, we obtain that

$$0 < |\nu\theta - a_\nu/q_\nu| \leqslant 1/q_\nu Q,\ (a_\nu, q_\nu) = 1,\ q/\log 2q \leqslant q_\nu \leqslant (\log x)^{10}\,. \tag{96}$$

In the sequel, we write $\beta_\nu := \nu\theta - a_\nu/q_\nu$.

Let $\eta := \frac{1}{2}(1-\delta)$, $x_2 := \exp(\log x)^\eta$. We plainly have

$$S^*(x;z,\nu\theta) = \sum_{k\leqslant x/x_2} z^{\Omega(k)}\left|\int_{x_2}^{x/k} \mathrm{d}T^*(u;z,k\nu\theta) + T^*(x_2;z,k\nu\theta)\right| + \sum_{x/x_2<k\leqslant x} z^{\Omega(k)}\,|T^*(x/k;z,k\nu\theta)|\,.$$

Using the trivial bounds

$T^*(x_2;z,k\nu\theta) \ll x_2(\log x_2)^{z-1}$ and $T^*(x/k;z,k\nu\theta) \ll (x/k)\big(\log(x/k)\big)^{z-1}$

for $x/x_2 < k \leqslant x$, and noting that

$$\mathrm{d}T^*(u;z,k\nu\theta) = e(k\beta_\nu u)\,\mathrm{d}T^*(u;z,ka_\nu/q_\nu)\,,$$

we arrive at

$$\begin{aligned} S^*(x;z,\nu\theta) \ll{}& x(\log x)^{(1+\eta)z-1} \\ &+ \sum_{k\leqslant x/x_2} z^{\Omega(k)}\left|\int_{x_2}^{x/k} e(k\beta_\nu u)\,\mathrm{d}T^*(u;z,ka_\nu/q_\nu)\right|. \end{aligned} \tag{97}$$

For each $k \leqslant x/x_2$, we put $q_\nu(k) := q_\nu/(q_\nu,k)$, $a_\nu(k) := ka_\nu/(q_\nu,k)$. Then $T^*(u;z,ka_\nu/q_\nu) = T^*(u;z,;a_\nu(k)/q_\nu(k))$ and we may apply Lemma 6 with $A = 10/\eta$ to write, whenever $x_2 \leqslant u \leqslant x$,

$$T^*(u;z,ka_\nu/q_\nu) = M(u) + R(u)$$

with

$$M(u) := \sum_{t\,|\,q_\nu(k)} \frac{\mu(t)z^{\Omega(q_\nu(k)/t)}}{\varphi(t)} H(tu/q_\nu(k);z;t),$$

$$R(u) \ll u\exp\{-c_0(\log x)^{\eta/3}\}\,.$$

The contribution of R to the integral in (97) may be estimated by partial summation. We have

$$\int_{x_2}^{x/k} e(k\beta_\nu u)\,\mathrm{d}R(u) \ll \frac{x}{k}\exp\{-c_0(\log x)^{\eta/3}\}(1+|\beta_\nu x|)$$
$$\ll \frac{x}{k}\exp\{-c_1(\log x)^{\eta/3}\}$$

for a suitable positive constant c_1, since $|\beta_\nu|x \leqslant (\log x)^{11}$ by (96). Thus the total contribution of the remainder term R to the right-hand side of (97) is

$$\ll x\exp\{-c_1(\log x)^{\eta/3}\}\sum_{k\leqslant x/x_2} z^{\Omega(k)}/k \ll x/\log x\,.$$

We estimate the contribution of the main term $M(u)$ to the integral of (97) by considering $M(u)$ as a double summation and bounding all the summands in absolute value. Moreover, we may also delete in this process the coprimality conditions appearing in the H-functions. In other words, we use the inequality between Stieltjes measures

$$|dM(u)| \leqslant \sum_{t\,|\,q_\nu(k)} \frac{1}{\varphi(t)}\,\mathrm{d}H(tu/q_\nu(k);z,1)\,. \tag{98}$$

Therefore we obtain

$$\left|\int_{x_2}^{x/k} e(k\beta_\nu u)\,\mathrm{d}M(u)\right| \leqslant \sum_{t\,|\,q_\nu(k)}\frac{1}{\varphi(t)}\int_0^{x/k}\mathrm{d}H(tu/q_\nu(k);z,1)$$
$$= \sum_{t\,|\,q_\nu(k)}\frac{1}{\varphi(t)}\sum_{n\leqslant tx/kq_\nu(k)} z^{\Omega(n)}$$
$$\ll \sum_{t\,|\,q_\nu(k)}\frac{1}{\varphi(t)}\,\frac{tx}{kq_\nu(k)}\left(\log\frac{x}{k}\right)^{z-1}$$
$$\ll \frac{\tau(q)(\log 2q)^2}{q}\,(k,q)\,\frac{x}{k}\left(\log\frac{x}{k}\right)^{z-1},$$

by (84) and (96). In the last stage, we have used the bound $t/\varphi(t) \leqslant q/\varphi(q) \ll \log 2q$ for all $t\,|\,q_\nu(k)$. Since $\tau(q)(\log 2q)^2 \ll q^\eta$, we see that the total contribution of the main term $M(u)$ to the right-hand side of (97) is

$$\ll xq^{-1+\eta}\sum_{k\leqslant x/x_2}(k,q)\,\frac{z^{\Omega(k)}}{k}\left(\log\frac{x}{k}\right)^{z-1}$$
$$\ll xq^{-1+\eta}\sum_{d\,|\,q} z^{\Omega(d)}\sum_{l\leqslant x/x_2 d}\frac{z^{\Omega(l)}}{l}\left(\log\frac{x}{ld}\right)^{z-1}$$
$$\ll xq^{-1+\eta}\tau(q)(\log x)^{2z-1} \ll q^{-\delta}x(\log x)^{2z-1}\,.$$

Inserting this into (97), we obtain the required estimate and this finishes the proof of the lemma.

Completion of the proof of Theorem 12. We want to apply (81) and hence need an upper bound for $S\left(x; \frac{1}{4}y, \nu\theta\right)$. We select $T = \log 2q^*(x; y, \theta)$. Since $q^*(x; y, \theta) \leqslant q(x; \theta)$ and $q^*(x; y, \theta) \ll (\log x)^{y/4}$, we infer from Lemmas 4 and 7 that we have uniformly for $1 \leqslant \nu \leqslant T$, $1 \leqslant y \leqslant 4$,

$$S^*\left(x; \tfrac{1}{4}y, \nu\theta\right) \ll x(\log x)^{y/2-1} q^*(x; y, \theta)^{-\delta}. \tag{99}$$

Now

$$\begin{aligned} S\left(x; \tfrac{1}{4}y, \nu\theta\right) &= \sum_{k \leqslant x} \frac{\left(\frac{1}{4}y\right)^{\Omega(k)}}{k} \left| \int_{1-}^{x/k} \frac{1}{u} \, \mathrm{d}T^*\left(u; \tfrac{1}{4}y, k\nu\theta\right) \right| \\ &= \sum_{k \leqslant x} \frac{\left(\frac{1}{4}y\right)^{\Omega(k)}}{k} \left| \frac{k}{x} T^*\left(x/k; \tfrac{1}{4}y, k\nu\theta\right) + \int_1^{x/k} \frac{1}{u^2} T^*\left(u; \tfrac{1}{4}y, k\nu\theta\right) \mathrm{d}u \right| \\ &= \sum_{k \leqslant x} \left(\tfrac{1}{4}y\right)^{\Omega(k)} \left| \frac{1}{x} T^*\left(x/k; \tfrac{1}{4}y, k\nu\theta\right) + \int_1^{x} \frac{1}{u^2} T^*\left(u/k; \tfrac{1}{4}y, k\nu\theta\right) \mathrm{d}u \right| \\ &\leqslant \frac{1}{x} S^*\left(x; \tfrac{1}{4}y, \nu\theta\right) + \int_1^x \frac{1}{u^2} S^*\left(u; \tfrac{1}{4}y, \nu\theta\right) \mathrm{d}u . \end{aligned}$$

By our monotonicity assumptions on $q^*(x; y, \theta)$ and (99), we have for $1 \leqslant u \leqslant x$

$$S^*\left(u; \tfrac{1}{4}y, \nu\theta\right) \ll u(\log 2u)^{(2-\delta)y/4-1} \{(\log x)^{y/4}/q^*(x; y, \theta)\}^{\delta} .$$

Inserting this into the previous bound, we obtain

$$S\left(x; \tfrac{1}{4}y, \nu\theta\right) \ll (\log x)^{y/2}/q^*(x; y, \theta)^{\delta} . \tag{100}$$

It follows that, with the value of T given above, we may take in (81), for all ν with $1 \leqslant \nu \leqslant T$,

$$\varepsilon_\nu^+(x, y; h_\theta) := q^*(x; y, \theta)^{-\delta} . \tag{101}$$

This is clearly a non-increasing function of x and $\varepsilon_\nu^+(x, y; h_\theta)(\log x)^{y/2}$ is plainly non-decreasing. Therefore we obtain

$$\sum_{n \leqslant x} \left(\frac{y}{4}\right)^{\Omega(n)} \frac{\Delta(n; h_\theta)^2}{n} \ll (\log x)^y q^*(x; y, \theta)^{-\delta} \{\log 2q^*(x; y, \theta)\}^2 .$$

Altering the value of δ, the factor $\{\log 2q^*(x; y, \theta)\}^2$ may be deleted. This yields (79) and finishes the proof of Theorem 12.