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Inserting this into (105), we arrive at

$$(\log x)^{-1} \sum_{n \leqslant x} \frac{\Delta(n; \theta\Omega)}{n 2^{\Omega(n)}} \ll \frac{1}{T} + \frac{\log q}{\log_2 x} \ll q_1(x; \theta)^{-1+o(1)} .$$

Since $q_1(x; \theta)$ is a non-decreasing function of x, this implies (107) and the proof is thereby completed.

Proof of Corollary 11. Put $\tau(n, \theta) := \sum_{d \mid n} d^{i\theta}$. When $f = \log$, we have that $g_\nu(n) = \tau(n, 2\pi\nu)$. By lemma 30.2 of [14] we infer that, uniformly for $1 \leqslant |\theta| \leqslant \exp\sqrt{\log x}$,

$$\sum_{p \leqslant x} \frac{|\tau(p, \theta)|}{p} = \sum_{p \leqslant x} \frac{|\cos\left(\frac{1}{2}\theta \log p\right)|}{p} = \frac{2}{\pi} \log x + O(1) .$$

This is proved by partial summation from a strong form of the prime number theorem. Thus we obtain that we have uniformly for $1 \leqslant \nu \leqslant \log x$

$$C_\nu(x; \log) = (1 - 2/\pi) \log_2 x + O(1) .$$

Inserting this into (105) with $T = \log x$ and choosing optimally $y = \pi/2$, we obtain

$$\Delta(n; f) < \xi(n) (\log_2 n) \left(\frac{4}{\pi}\right)^{\Omega(n)} \quad \mathrm{pp}l$$

for all $\xi(n) \to \infty$. This implies the required result by a now standard device.

6. Metric results

In this last section, we investigate the problem of uniform distribution on divisors from a further statistical point of view, regarding as random not only the integer n but also the function f. Thus, we define a measure μ on the set $\mathbf{A}$ of all real valued arithmetical function as the inverse image of the Haar measure on the compact group $(\mathbf{R}/\mathbf{Z})^{\mathrm{N}}$ by the canonical mapping $f \mapsto \langle f \rangle$. In other words, μ is characterised by the property that for all finite families $\{E_j : 1 \leqslant j \leqslant k\}$ of measurable subsets of the torus $\mathbf{R}/\mathbf{Z}$ and for all integers $n_1, n_2, \ldots, n_k$, we have

$$\mu\{f \in \mathbf{A} : \langle f(n_j) \rangle \in E_j \ (1 \leqslant j \leqslant k)\} = \prod_{j=1}^{k} \lambda(E_j) ,$$

where λ stands for the Lebesgue measure on $\mathbf{R}/\mathbf{Z}$. The only basic property of μ that we shall use is the orthogonality relation

$$(113) \qquad \int_{\mathrm{A}} e(\nu f(n) - \nu f(m))\,\mathrm{d}\mu(f) = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n\,. \end{cases}$$

If an arithmetic function behaves statistically, then one expects that the Weyl sums

$$g_\nu(n) := \sum_{d \mid n} e(\nu f(d))\,,$$

and hence the discrepancy $\Delta(n; f)$, will normally have size roughly $\sqrt{\tau(n)}$. The purpose of the next theorem is to establish that this is indeed the case.

THEOREM 14. *Let* $\xi(n) \to \infty$. *For* μ *-almost all arithmetic functions* f, *we have*

$$(114) \qquad \Delta(n; f) < \xi(n)(\log_2 n)^3 (\tau(n))^{1/2} \quad \text{pp}l\,.$$

Moreover, the exponent $\frac{1}{2}$ *is sharp in this statement.*

Proof. The upper bound follows from (24) with $y = 2$, $T = (\log x)^2$, namely

$$(115) \qquad \begin{aligned} S(x; f) &:= \frac{1}{\log x} \sum_{n \leqslant x} \frac{\Delta(n; f)^2}{2^{\Omega(n)} n} \\ &\ll \frac{1}{(\log x)^3} + \frac{\log_2 x}{\sqrt{(\log x)}} \sum_{1 \leqslant \nu \leqslant T} \frac{1}{\nu} H_\nu(x; f)\,, \end{aligned}$$

with

$$H_\nu(x; f) := \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{\Omega(k)} \frac{1}{k^{1+\sigma}} \left| \sum_{m=1}^{\infty} \frac{e(\nu f(km))}{m^{1+\sigma} 2^{\Omega(m)}} \right|^2 \quad (\sigma := 1/\log x)\,.$$

We have $H_\nu(x; f) \leqslant 2H_\nu^\dagger(x; f) + 2H_\nu^\ddagger(x; f)$ where the m-sum is restricted to $m \leqslant (\log x)^3$ in $H_\nu^\dagger$ and to $m > (\log x)^3$ in $H_\nu^\ddagger$. We note right away the trivial estimate

$$H_\nu^\dagger(x; f) \ll \sqrt{\log x}\, \log_2 x\,,$$

which follows from (6) by partial summation. We deduce from this and (115) that

$$(116) \qquad S(x; f) \ll (\log_2 x)^3 + R(x; f)\,,$$

with

$$R(x; f) := \frac{\log_2 x}{\sqrt{\log x}} \sum_{1 \leqslant \nu \leqslant T} \frac{1}{\nu} H_\nu^\ddagger(x; f)\,.$$

Expanding the square and integrating over f with respect to μ we get from (113) that

$$\int_{\mathbf{A}} H_\nu^\dagger(x; f)\,\mathrm{d}\mu(f) = \sum_{k=1}^{\infty} \frac{1}{2^{\Omega(k)} k^{1+\sigma}} \sum_{m > (\log x)^3} \frac{1}{4^{\Omega(m)} m^{2+2\sigma}} \ll 1/(\log x)^2 .$$

Hence

$$\int_{\mathbf{A}} R(x; f)\,\mathrm{d}\mu(f) \ll \frac{(\log_2 x)^2}{(\log x)^{5/2}}$$

Markov's inequality thus implies that

$$\mu\{f \in \mathbf{A} : R(2^l; f) \geqslant 1\} \ll 1/l^2 \qquad (l = 1, 2, \ldots) ,$$

so it follows by the Borel-Cantelli theorem, that for μ-almost all f we have

$$R(2^l; f) \ll 1 \qquad (l = 1, 2, \ldots) .$$

In view of (116), we see that the estimate $S(2^l; f) \ll (\log 2l)^3$ holds μ-almost surely in f and uniformly for $l \geqslant 1$. However, using the trivial bound $\Delta(n; f) \leqslant \tau(n)$, we readily see that

$$S(x; f) - S(2^l; f) \ll 1 \qquad (2^l \leqslant x < 2^{l+1}) .$$

This yields that for μ-almost all f and uniformly in $x \geqslant 3$, we have

$$S(x; f) \ll (\log_2 x)^3 ,$$

which in turn implies (114).

To show that the exponent $\frac{1}{2}$ is sharp, we simply use (109) with $\nu = 1$ in the form

$$4\pi^2 \int_{\mathbf{A}} \Delta(n; f)^2 \mathrm{d}\mu(f) \geqslant \int_{\mathbf{A}} |g_1(n)|^2 \mathrm{d}\mu(f) = \tau(n) ,$$

where the equality follows from (113). This plainly implies that there is no $\alpha < \frac{1}{2}$ such that $\Delta(n; f) \ll \tau(n)^\alpha$ pp$l$ for μ-almost all f: such a bound is actually false as soon as $\tau(n)$ is large enough.

The same quadratic mean approach that we used for Theorem 14 yields metric results for more restricted classes of arithmetic functions. We quote without proof the following theorem.

THEOREM 15. *The function* $d \mapsto \theta^d$ *is* erd *for almost all* $\theta > 1$ *and the function* $d \mapsto \lambda\theta^d$ *is* erd *for all* $\theta > 1$ *and almost all* λ.

More precisely, the corresponding discrepancies satisfy

(117) $$\Delta(n;f) \ll \tau(n)^{1/2+o(1)} \quad \text{pp}l\,,$$

under the indicated hypotheses, and the exponent $\frac{1}{2}$ *is sharp.*

Theorems 14 and 15 together provide an optimal strengthening of theorem 5 of Dupain, Hall & Tenenbaum [4].

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