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Each polyhedron $V(s)$ is homotopically equivalent to V. More precisely we have

PROPOSITION 2.3. *For every sufficiently large s the locally symmetric space V is homeomorphic to the interior of the polyhedron $V(s)$ in V, and $V(s)$ is a strong deformation retract of V.*

For the proof see [L3], Theorems 5.2 and 5.5.

3. ESTIMATES FOR THE BOUNDARY SUBPOLYHEDRA

We wish to apply Proposition 1.1 to the polyhedra $V(s)$ in the above exhaustion and then take the limit for $s \to \infty$. To that end we need estimates for the second fundamental forms and the volumes of the (lower dimensional) boundary polyhedra.

For each Siegel set $\mathcal{S}_i := q_i\mathcal{S}$ which is part of the fundamental set Ω we have its truncated part

$$\mathcal{S}_i(s) := \mathcal{S}_i - \bigcup_{\alpha\in\Delta} \left(\mathcal{B}_{i\alpha}(s) \cap \mathcal{S}_i\right).$$

The top dimensional boundary faces of $\mathcal{S}_i(s)$ in $\mathcal{S}_i$ (resp. of $\Omega(s)$ in Ω) are subsets of horospheres:

$$\mathcal{H}_{i\alpha}(s) := \{\tau_\alpha^{-1}\tilde{h}_{i\alpha} = -s\} \cap \mathcal{S}_i(s) \ , \qquad \alpha \in \Delta \ .$$

The "horospherical" pieces $\mathcal{H}_{i\alpha}(s)$ together with their Γ-translates form the boundary of the manifold with corners $X(s)$ in X. For any nonempty subset Θ of Δ we set

$$\mathcal{H}_{i\Theta}(s) := \bigcap_{\alpha\in\Theta} \mathcal{H}_{i\alpha}(s) \subset \mathcal{S}_i(s).$$

The various boundary subpolyhedra of $V(s)$ are then unions of projections of the pieces $\mathcal{H}_{i\Theta}(s)$ under the canonical projection $\pi : X \to V$. More precisely, as explained in Section 2, for any subset $\Theta \subset \Delta$, we have the equivalence relation on the set $I = \{1, \dots, m\}$

$$j \sim_\Theta l \quad \text{if and only if} \quad \Gamma q_j P_\Theta = \Gamma q_l P_\Theta$$

(the q_i are as in Proposition 2.1). This relation $\sim_\Theta$ induces a partition, $I(\Theta)$, of the set I whose components will be denoted by E. Let $n = \dim X = \dim V$, let k be the cardinality of Θ and let $E \in I(\Theta)$. Then $V_E^{n-k}(s) := \pi\big(\bigcup_{i\in E} \mathcal{H}_{i\Theta}(s)\big)$ is a $(n-k)$-dimensional boundary polyhedron of $V(s)$; and moreover, any boundary polyhedron arises in this way (see [L3] §4).

REMARK. The minimal possible dimension which occurs is $n-q$ where q is the $\mathbb{Q}$-rank of $\mathbf{G}$. It is also interesting to note (though not needed below) that the outer angles are isomorphic to $\mathbb{Q}$-Weyl chambers and their walls at infinity.

We shall use the following well-known fact about Jacobi fields in symmetric spaces (see [K] Theorem 2.2.9). A Jacobi field along a geodesic ray is called *stable* if its length is bounded.

LEMMA 3.1. *Let $r:[0,\infty)\to X$ be a unit-speed geodesic ray in the symmetric space X (of noncompact type). Set $p=r(0)$. Then the unique stable Jacobi field $J_u(s)$ along $r(s)$ with $J_u(0)=u\in T_pX$ can be written as*

$$J_u(s)=\sum_j e^{-\lambda_j s}a_jE_j(s)$$

where $\{E_j(s)\}$ is an orthonormal frame of parallel fields along r, the λ_j are non-negative (uniform) constants and $u=\sum_j a_jE_j(0)$.

LEMMA 3.2. *Let $s\geq 0$. The second fundamental forms of every boundary polyhedron $V_E^{n-k}(s)$ with respect to outer angles in $V(s)$ are uniformly bounded by a constant independent of E,k and s.*

Proof. Since the claim is local we can work in the universal covering space X. As we noted above the preimage of $V_E^{n-k}(s)$ in X under the projection π is the union of a *finite* number of horospherical sets

$$\mathcal{H}_{i\Theta}(s)=\bigcap_{\alpha\in\Theta}\mathcal{H}_{i\alpha}(s)\subset\bigcap_{\alpha\in\Theta}\{\tau_\alpha^{-1}\tilde{h}_{i\alpha}=-s\},$$

where Θ is a subset of Δ with k elements. The (inner) unit normal field of the horosphere $\{\tau_\alpha^{-1}\tilde{h}_{i\alpha}=-s\}$ is given by $Z_{i\alpha}:=-\text{grad}\ \tilde{h}_{i\alpha}$ (see e.g. [HI] Proposition 3.1). Using $d\pi$ any element in the outer angle $O(\pi(p))$ of $V_E^{n-k}(s)$ at a point $\pi(p)\in V_E^{n-k}(s)$ can then be identified with a *positive* linear combination (of norm 1) of the radial fields $Z_{i\alpha}(p)$, $\alpha\in\Theta$. It therefore suffices to show that for any pair (i,α) the second fundamental form of $V_E^{n-k}(s)$ relative to $d\pi Z_{i\alpha}$ is uniformly bounded. We fix i and α and write Z for $Z_{i\alpha}$. For $p\in X$ let $\langle .\,,.\rangle_p$ denote the Riemannian metric of X at p. Let $u,v\in T_pX$ be such that $d\pi(u),d\pi(v)\in T_{\pi(p)}V_E^{n-k}(s)$. Using the above identifications the second fundamental form of $V_E^{n-k}(s)\subset V(s)$ with respect to Z can be written as

$$\text{II}_Z(u,v)\,(p)=\langle D_uZ,v\rangle_p\,.$$

According to [HI], Proposition 3.1, we have $D_u Z(p) = J_u'(0)$ where J_u is the stable Jacobi field along the (unique) geodesic ray, say r, in X which joins p to $c_{i\alpha}(\infty) \in \partial_\infty X$ and with initial value $J_u(0) = u$. By Lemma 3.1 there are orthonormal parallel fields $E_j(s)$ along r and constants $\lambda_j \geq 0$ such that $J_u(s) = \sum_j e^{-\lambda_j s} a_j E_j(s)$ with $u = \sum_j a_j E_j(0)$. Consequently we get $J_u'(0) = -\sum_j \lambda_j a_j E_j(0)$ and finally, for $v = \sum_j b_j E_j(0)$, $|\mathrm{II}_Z(u,v)(p)| = \left|-\sum_j \lambda_j a_j b_j\right| \prec \|u\|\|v\|$. $\square$

We next estimate the volumes of the boundary polyhedra. Recall from Section 2.1 the parametrization of X by horocyclic coordinates

$$\mu \colon Y = U \times Z \times A \longmapsto X \ ; \ \left(u, \tau(m), a\right) \longmapsto uma \cdot x_0 \, .$$

Let dx^2 be the G-invariant Riemannian metric on X induced by the Cartan-Killing form of the Lie algebra $\mathfrak{g}$ of G and let dz^2 be the invariant metric on Z. Further let da^2 (resp. du^2) be the left-invariant metric on A (resp. U). Finally set $dy^2 := \mu^* dx^2$.

LEMMA 3.3. *Let dv_Y, dv_U, dv_Z and dv_A denote the volume elements of the metrics dy^2, du^2, dz^2 and da^2. Then at the point $(u,z,a) \in Y$ we have*

$$2^e dv_Y = \rho(a)^{-1} dv_U \wedge dv_Z \wedge dv_A$$

where $e = \frac{1}{2}\dim U$ and ρ is the sum of all positive roots (counted with multiplicity); it can be written in the form $\rho = \sum_{\alpha \in \Delta} c_\alpha \alpha$, $c_\alpha > 0$.

For the proof see [B2] Corollary 4.4.

LEMMA 3.4. *For the $(n-k)$-dimensional volume of each boundary polyhedron $V_E^{n-k}(s)$ of $V(s)$ one has the estimate*

$$\mathrm{Vol}\left(V_E^{n-k}(s)\right) \prec s^{q-k} e^{-cs} \, ,$$

where $q = \dim A$ is the $\mathbb{Q}$-rank of $\mathbf{G}$ and $c > 0$ is a constant (independent of E, k and s).

Proof. We again consider the preimage of $V_E^{n-k}(s)$ in X under the map π. We need to estimate the volume of each horospherical piece

$$\mathcal{H}_{i\Theta}(s) = \bigcap_{\alpha \in \Theta} \{\tau_\alpha^{-1} \tilde{h}_{i\alpha} = -s\} \cap \mathcal{S}_i(s), \quad i \in E \, .$$

It clearly suffices to carry out the estimates for $i = 1$; note that $q_1 = e$. For the horocyclic coordinate map $\mu : Y \to X$ and the canonical projection

$\pi_A : Y \to A$ we set $A_\Theta(s) := \pi_A \circ \mu^{-1}\big(\mathcal{H}_{1\Theta}(s)\big) \subset A$. The set $A_\Theta(s)$ is contained in an "affine" subspace of A of the form $a_1 a_*(s) A^{q-k}$ where $a_1 a_*(s) \in A$ and A^{q-k} is a $q-k$-dimensional subgroup of A (see Sections 3 and 4 of [L2]). We denote the restriction of dv_A to A^{q-k} by $dv_{A^{q-k}}$; for $k = q$ we have $A^0 = e$ and we set $dv_{A^0} \equiv 1$. By Lemma 3.3 we have (for k equal to the number of elements of Θ)

$$\mathrm{Vol}\big(V_E^{n-k}(s)\big) \prec \int_{\mu^{-1}(\mathcal{H}_{1\Theta}(s))} \rho(a)^{-1}\, dv_U \wedge dv_Z \wedge dv_{A^{q-k}}\,.$$

Since the horospherical piece $\mathcal{H}_{1\Theta}(s)$ is part of a Siegel set $\mathcal{S}_{\omega,\tau}$ with ω relatively compact (and hence of finite volume) in UM we get

$$\int_{\mu^{-1}(\mathcal{H}_{1\Theta}(s))} \rho(a)^{-1} dv_U \wedge dv_Z \wedge dv_{A^{q-k}} \prec$$

$$\prec \int_\omega dv_U \wedge dv_Z \int_{A_\Theta(s)} \rho(a)^{-1} dv_{A^{q-k}} \prec \int_{A_\Theta(s)} \rho(a)^{-1} dv_{A^{q-k}}\,.$$

Also by definition of a Siegel set we have $\alpha(a) \geq \tau \succ 1$ for all $\alpha \in \Delta$. Moreover, the computations in the proof of Lemma 4.1 (and Lemma 3.5) in [L2] show that for all $\alpha \in \Theta$ one has $\alpha\big(a_1 a_*(s)\big) \succ e^{\mu_\alpha s}$ with $\mu_\alpha > 0$. Hence, as $\Theta \subset \Delta$ is not empty and since $\rho = \sum_{\alpha\in\Delta} c_\alpha \alpha\, (c_\alpha > 0)$, there is a uniform constant $c > 0$ such that $\rho(a)^{-1} \prec e^{-cs}$ for all $a \in A_\Theta(s)$. As noted above the set $A_\Theta(s)$ is contained in a $(q-k)$-dimensional affine cone in A. It is similar (in the sense of Euclidean geometry) to $A_\Theta(0)$ with similarity factor s (see the proof of Lemma 4.1 in [L2]). Hence we eventually get $\int_{A_\Theta(s)} dv_{A^{q-k}} \prec s^{q-k}$ and the Lemma follows. □

4. A NEW PROOF OF THE GAUSS-BONNET FORMULA

In this section we present a new simplified proof of the Gauss-Bonnet theorem for higher rank locally symmetric spaces.

THEOREM 4.1. *Let X be a Riemannian symmetric space of noncompact type and $\mathbb{R}$-rank ≥ 2 and let Γ be an irreducible, torsion-free (non-uniform) lattice in the group of isometries of X. Then for the locally symmetric space $V = \Gamma\backslash X$ the Gauss-Bonnet formula holds:*

$$\chi(V) = \int_V \Psi\, dv\,.$$