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**Artikel:** INTRODUCTORY NOTES ON RICHARD THOMPSON'S GROUPS
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THEOREM 4.10. *Thompson's group $F$ is not an elementary amenable group.*

*Proof.* According to (a) of Chou's Proposition 2.2, it suffices to prove that $F \notin EG_\alpha$ for every ordinal $\alpha$. Since $EG_0$ consists of finite groups and Abelian groups, it is clear that $F \notin EG_0$, so assume that $\alpha > 0$ and that $F \notin EG_\beta$ for every ordinal $\beta < \alpha$.

If $\alpha$ is a limit ordinal, then there is nothing to prove. Suppose that $\alpha$ is not a limit ordinal. It must be shown that $F$ cannot be constructed from groups in $EG_{\alpha-1}$ as a group extension or as a direct union.

First consider group extensions. Suppose that $F$ contains a normal subgroup $N$ such that $N$, $F/N \in EG_{\alpha-1}$. Since $F \notin EG_{\alpha-1}$, $N$ is nontrivial. Theorem 4.3 implies that $[F,F] \subset N$. Now Theorem 4.1 and Lemma 4.4 easily imply that $N$ contains a subgroup isomorphic with $F$. Proposition 2.1 of [C] states that subgroups of groups in $EG_{\alpha-1}$ are also in $EG_{\alpha-1}$. Thus $F \in EG_{\alpha-1}$, contrary to hypothesis. This proves that $F$ cannot be constructed from $EG_{\alpha-1}$ as a group extension.

Second consider direct unions. Suppose that $F$ is a direct union of groups in $EG_{\alpha-1}$. This is clearly impossible because $F$ is finitely generated.

This proves Theorem 4.10. □

We next show that $F$ is a totally ordered group (this also follows from [BriS]). Define the set of *order positive* elements of $F$ to be the set $P$ of functions $f \in F$ such that there exists a subinterval $[a,b]$ of $[0,1]$ on which the derivative of $f$ is less than 1 and $f(x) = x$ for $0 \leq x \leq a$. It is easy to see that the positive elements of $F$ are indeed order positive. It is clear that $F = P^{-1} \cup \{1\} \cup P$. It is easy to see that $P$ is closed under multiplication and $f^{-1}Pf \subset P$ for every $f \in F$. This proves Theorem 4.11.

THEOREM 4.11. *Thompson's group $F$ is a totally ordered group.*

## §5. THOMPSON'S GROUP $T$

The material in this section is mainly from unpublished notes of Thompson [T1].

Consider $S^1$ as the interval $[0,1]$ with the endpoints identified. Then $T$ is the group of piecewise linear homeomorphisms from $S^1$ to itself that map images of dyadic rational numbers to images of dyadic rational numbers and that are differentiable except at finitely many images of dyadic rational numbers and on intervals of differentiability the derivatives are powers of 2. Just as we proved that $F$ is a group, it is easy to see that $T$ is indeed a group.

While $T$ is defined as a group of piecewise linear homeomorphisms of $S^1$, Ghys and Sergiescu [GhS] proved that there is a homeomorphism of $S^1$ that conjugates it to a group of $C^\infty$ diffeomorphisms. (Thurston had proved earlier that $T$ has a representation as a group of $C^\infty$ diffeomorphisms of $S^1$.)

EXAMPLE 5.1. The elements $A$ and $B$ of $F$ induce elements of $T$, which will still be denoted by $A$ and $B$. A third element of $T$ is the function $C$ defined (on $[0,1]$) by

$$C(x) = \begin{cases} \frac{x}{2} + \frac{3}{4}, & 0 \leq x \leq \frac{1}{2} \\ 2x - 1, & \frac{1}{2} \leq x \leq \frac{3}{4} \\ x - \frac{1}{4}, & \frac{3}{4} \leq x \leq 1. \end{cases}$$

We can associate tree diagrams and unique reduced tree diagrams to elements of $T$ almost exactly as we did to elements of $F$. The only difference is the following. Elements of $F$ map leftmost leaves of domain trees to leftmost leaves of range trees. When an element of $T$ does not do this, we denote the image in its range tree of the leftmost leaf of its domain tree with a small circle. For example, the reduced tree diagram for $C$ is in Figure 11.

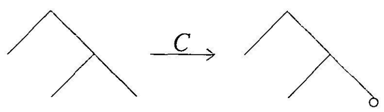


FIGURE 11
The reduced tree diagram for $C$

LEMMA 5.2. *The elements $A$, $B$, and $C$ generate $T$ and satisfy the following relations:*

1) $[AB^{-1}, A^{-1}BA] = 1$,
2) $[AB^{-1}, A^{-2}BA^2] = 1$,
3) $C = B(A^{-1}CB)$,
4) $(A^{-1}CB)(A^{-1}BA) = B(A^{-2}CB^2)$,
5) $CA = (A^{-1}CB)^2$, *and*
6) $C^3 = 1$.

*Proof.* Let $H$ be the subgroup of $T$ generated by $\{A, B, C\}$. Since $\{A, B\}$ is a generating set for $F$, $F \subset H$. Suppose $f \in T$. Let $[x] = f([0])$. If $[x] = [0]$, then $f \in F$ and hence $f \in H$. If $[x] \neq [0]$, then there is an element $h \in F$ with $h(x) = \frac{3}{4}$ by Lemma 4.2. Then $g = C^{-1}hf$ fixes $[0]$, so $g \in F$. Hence $f = h^{-1}Cg \in H$ and $H = T$. Thus $A$, $B$, and $C$ generate $T$.

Relations 1) and 2) are proved in Section 3.

Consider relation 3). It is equivalent to the relation $CBC^{-1} = AB^{-1}$. The reduced tree diagram for $CBC^{-1}$ is computed in Figure 12, the notation being straightforward.

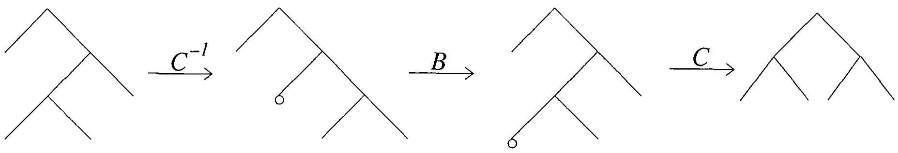


FIGURE 12

Computing the reduced tree diagram for $CBC^{-1}$

Referring to Figure 1 shows that $AB^{-1}$ has the same reduced tree diagram as $CBC^{-1}$, which completes the verification of relation 3).

Consider relation 4). It is equivalent to

$$(B^{-1}C^{-1}A)(AB^{-1})(A^{-1}CB) = BA^{-1}B^{-1}A,$$

where the term $A^{-1}CB$ here corresponds to the same term in relation 4). We compute a tree diagram for the left side of this equation in Figure 13.

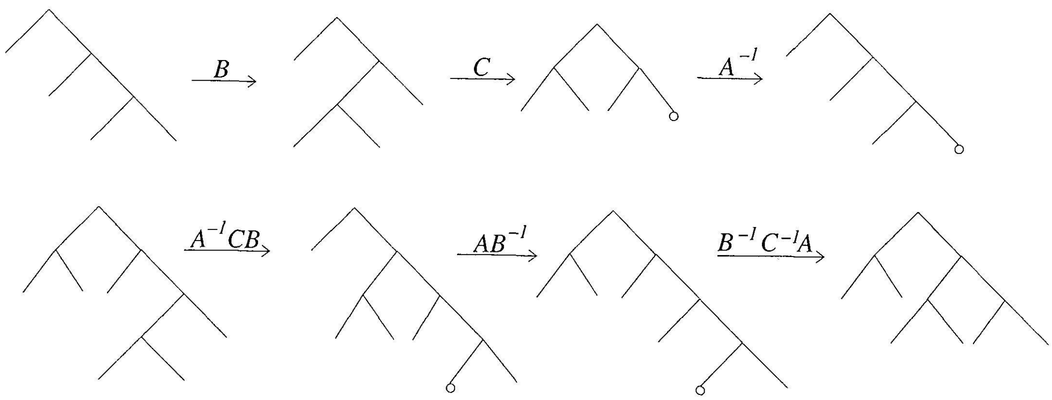


FIGURE 13

Computing a tree diagram for $(B^{-1}C^{-1}A)(AB^{-1})(A^{-1}CB)$

Referring to Figure 1 now completes the verification of relation 4).

Relation 6) is easily verified using the reduced tree diagram for $C$.

Finally consider relation 5). Use relation 6) and then relation 3) to rewrite relation 5): $CA = A^{-1}CBA^{-1}CB \Leftrightarrow CA = A^{-1}C^{-1}(C^{-1}BA^{-1}CB) \Leftrightarrow CA = A^{-1}C^{-1} \Leftrightarrow (AC)^2 = 1$. The reduced tree diagram for $AC$ is computed in Figure 14.

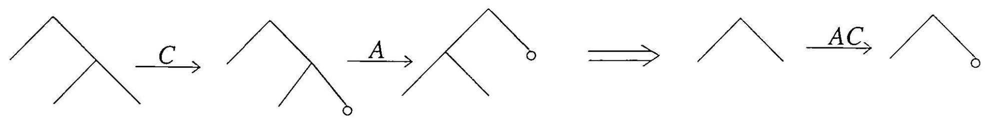


FIGURE 14
Computing the reduced tree diagram for $AC$

Hence $AC$ acts on $S^1$ by translation by $\left[\frac{1}{2}\right]$, and so $(AC)^2 = 1$, which gives relation 5). □

Let

$$T_1 = \langle A, B, C : [AB^{-1}, A^{-1}BA], [AB^{-1}, A^{-2}BA^2], C^{-1}B(A^{-1}CB), \\ \big((A^{-1}CB)(A^{-1}BA)\big)^{-1}B(A^{-2}CB^2), (CA)^{-1}(A^{-1}CB)^2, C^3 \rangle .$$

LEMMA 5.3. *There is a surjection $T_1 \to T$ that maps the formal symbols $A$, $B$, and $C$ to the functions $A$, $B$, and $C$ in $T$.*

*Proof.* This follows immediately since the functions $A$, $B$, and $C$ satisfy the relations 1) - 6). □

LEMMA 5.4. *The subgroup of $T_1$ generated by $A$ and $B$ is isomorphic to $F$.*

*Proof.* The results of Section 3 show that there exists a group homomorphism from $F$ to the subgroup of $T_1$ generated by $A$ and $B$ whose composition with the map from $T_1$ to $T$ is the identity map on $F$. This proves Lemma 5.4. □

It is easier at this point to prove that $T$ is simple than to prove that $T_1$ is simple. However, it is preferable to prove that $T_1$ is simple, since then Lemma 5.3 implies that $T$ is isomorphic to $T_1$.

Define the elements $X_n$, $n \geq 0$, of $T_1$ by $X_0 = A$ and $X_n = A^{-(n-1)}BA^{n-1}$ for $n \geq 1$. It follows from Theorem 3.4 and Lemma 5.4 that $X_nX_k = X_kX_{n+1}$ if $k < n$. Define the elements $C_n$, $n \geq 1$, of $T_1$ by $C_n = A^{-(n-1)}CB^{n-1}$. For convenience we define $C_0 = 1$.

To gain some insight into these elements $C_n$, in Figure 15 we calculate reduced tree diagrams for the corresponding elements, still called $C_n$, in $T$. The reduced tree diagram for $C_1$ is given in Figure 11, and the reduced tree diagram for $C_2$ is given in Figure 13. This calculation shows that $C_n$ permutes the images of the $n+2$ intervals

$$[0, 1-2^{-1}], [1-2^{-1}, 1-2^{-2}],$$
$$[1-2^{-2}, 1-2^{-3}], \ldots, [1-2^{-n}, 1-2^{-(n+1)}],\ [1-2^{-(n+1)}, 1]$$

cyclically.

The rest of this section deals with the group $T_1$.

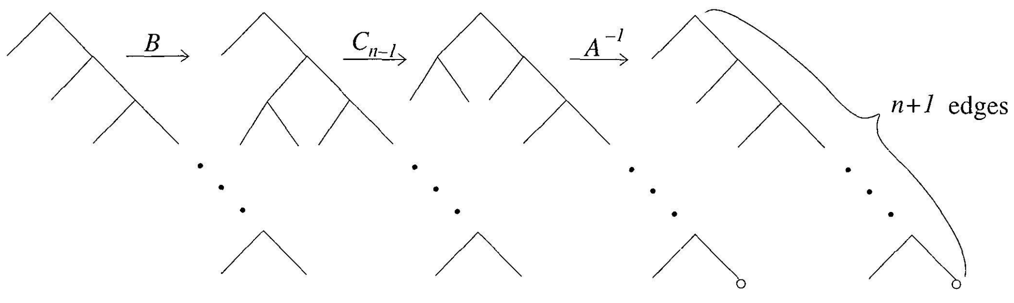


FIGURE 15

Inductively computing the reduced tree diagram for $C_n$ with $n \geq 3$

LEMMA 5.5. *If $k$, $n$ are positive integers and $k \leq n$, then*

*i)* $C_n = X_n C_{n+1}$,

*ii)* $C_n X_k = X_{k-1} C_{n+1}$, *and*

*iii)* $C_n A = C_{n+1}^2$.

*Proof.*

$$C_n = A^{-(n-1)} C B^{n-1} = A^{-(n-1)} B (A^{-1} C B) B^{n-1}$$
$$= (A^{-(n-1)} B A^{n-1}) (A^{-n} C B^n) = X_n C_{n+1},$$

which proves i).

If $k = 1$, ii) follows from the definition. If $k = n = 2$, ii) follows from relation 4). If $k = 2$ and $n > 2$, then by induction on $n$

$$C_n X_2 \overset{i)}{=} X_{n-1}^{-1} C_{n-1} X_2 \overset{\text{induct}}{=} X_{n-1}^{-1} X_1 C_n = X_1 X_n^{-1} C_n \overset{i)}{=} X_1 C_{n+1}.$$

If $k \geq 3$, then by induction on $k$

$$C_n X_k = A^{-1} C_{n-1} B X_k$$
$$= A^{-1} C_{n-1} X_{k-1} B \overset{\text{induct}}{=} A^{-1} X_{k-2} C_n B = (A^{-1} X_{k-2} A)(A^{-1} C_n B)$$
$$= X_{k-1} C_{n+1}.$$

Equation iii) follows by induction on $n$. If $n = 1$ then it is relation 5). If $n > 1$, then

$$C_nA = A^{-1}C_{n-1}BA = A^{-1}C_{n-1}AX_2$$
$$\stackrel{\text{induct}}{=} A^{-1}C_n^2X_2 \stackrel{ii)}{=} A^{-1}C_nBC_{n+1} = C_{n+1}^2 . \qquad \square$$

LEMMA 5.6. *If $n$ is a positive integer, $m \in \{1, \ldots, n+1\}$, and $r, s \in \{0, \ldots, n\}$, then*

*i)*

$$C_n^m X_r = \begin{cases} X_{r-m} C_{n+1}^m, & r \geq m \\ C_{n+1}^{m+1}, & r = m-1 \\ X_{r+(n+2-m)} C_{n+1}^{m+1}, & r < m-1 ; \end{cases}$$

*ii)*

$$X_s^{-1} C_n^m = \begin{cases} C_{n+1}^{m+1} X_{(s+m)-(n+2)}^{-1}, & s \geq (n+2)-m \\ C_{n+1}^m, & s = (n+1)-m \\ C_{n+1}^m X_{s+m}^{-1}, & s \leq n-m ; \end{cases}$$

*iii)* $C_n^m = X_{(n+1)-m} C_{n+1}^m$ *;*

*iv)* $C_n^m = C_{n+1}^{m+1} X_{m-1}^{-1}$ *;*

*v)* $C_n^{n+2} = 1$.

*Proof.* The first line of i) follows from Lemma 5.5.ii). If $r = 0$, the second line is Lemma 5.5.iii); if $r > 0$

$$C_n^m X_r = C_n C_n^r X_r \stackrel{5.5.ii)}{=} C_n A C_{n+1}^r \stackrel{5.5.iii)}{=} C_{n+1}^2 C_{n+1}^r = C_{n+1}^{m+1} .$$

This proves i) if $r \geq m-1$.

$$C_n^m = C_n^{m-1} C_n \stackrel{5.5.i)}{=} C_n^{m-1} X_n C_{n+1} \stackrel{5.5.ii)}{=} X_{n-(m-1)} C_{n+1}^m ,$$

which proves iii). If $r < m-1$, then

$$C_n^m X_r = C_n^{m-(r+1)} C_n^{r+1} X_r \stackrel{5.5}{=} C_n^{m-(r+1)} C_{n+1}^{r+2}$$
$$\stackrel{iii)}{=} X_{n+1-(m-(r+1))} C_{n+1}^{m-(r+1)} C_{n+1}^{r+2} = X_{r+(n+2-m)} C_{n+1}^{m+1} ,$$

which finishes the proof for i).

The first line of ii) follows from i), with $s = r+(n+2-m)$. The second line of ii) follows from iii). The third line of ii) follows from i), with $s = r - m$. Equation iv) follows from the second line of i). If $t \geq 1$, then

$$C_t^{t+2} = C_t C_t^{t+1} \stackrel{iii)}{=} C_t A C_{t+1}^{t+1} \stackrel{5.5.iii)}{=} C_{t+1}^2 C_{t+1}^{t+1} = C_{t+1}^{t+3} .$$

Since $C_1^3 = C^3 = 1$, this proves v) and completes the proof of Lemma 5.6. $\square$

Following the terminology for $F$, an element of $T_1$ which is a product of nonnegative powers of the $X_i$'s will be called *positive* and an inverse of a positive element will be called *negative*.

THEOREM 5.7. *If $g \in T_1$, then $g = pC_n^m q^{-1}$ for some positive elements $p$, $q$ and nonnegative integers $m$, $n$ with $m < n+2$.*

*Proof.* We first show that if $i$, $j$, $k$, and $l$ are positive integers, then there are positive elements $p$ and $q$ and nonnegative integers $m$ and $n$ such that $C_j^i C_l^k = pC_n^m q^{-1}$. Suppose that $i$, $j$, $k$, and $l$ are positive integers and that $g = C_j^i C_l^k$. Since $C_j^{j+2} = C_l^{l+2} = 1$ by Lemma 5.6.v), we can assume that $i < j+2$ and $k < l+2$. Let $n \geq \max\{j, l\}$. By Lemma 5.6.iii) and Lemma 5.6.iv), there is a positive integer $r$ and there are positive elements $p$ and $q$ such that $C_j^i = pC_n^i$ and $C_l^k = C_n^r q^{-1}$. Hence $C_j^i C_l^k = pC_n^{i+r} q^{-1}$.

Let $H = \{g \in T_1 : g = pC_n^m q^{-1}$ for some positive elements $p$, $q$, and nonnegative integers $m$, $n$ with $m < n+2\}$. Lemma 5.6.v) easily implies that $H$ is closed under inversion. To show that $H$ is closed under multiplication, suppose that $g_1, g_2 \in H$. Then $g_1 = p_1 C_j^i q_1^{-1}$ and $g_2 = p_2 C_l^k q_2^{-1}$ for some positive elements $p_1$, $p_2$, $q_1$, and $q_2$ and some nonnegative integers $i$, $j$, $k$, and $l$ with $i < j+2$ and $k < l+2$. By Corollary 2.7, there are positive elements $p_3$ and $q_3$ such that $q_1^{-1} p_2 = p_3 q_3^{-1}$. Hence $g_1 g_2 = p_1 C_j^i q_1^{-1} p_2 C_l^k q_2^{-1} = p_1 C_j^i p_3 q_3^{-1} C_l^k q_2^{-1}$. Lemma 5.6.iii) and Lemma 2.8, which states that the set of positive elements of $F$ is closed under multiplication, show that if $i > 0$ and $j > 0$, then we may replace $C_j^i$ by $C_{j+1}^i$. Hence we may assume that if $i > 0$, $j > 0$, and $X_r$ occurs in $p_3$, then $j \geq r$. We may likewise assume that if $k > 0$, $l > 0$, and $X_s$ occurs in $q_3$, then $l \geq s$. Now Lemmas 5.6.i), 5.6.ii), and 2.8 show that there are positive elements $p_4$ and $q_4$ and nonnegative integers $r$, $s$, $t$, and $u$ such that $g_1 g_2 = p_4 C_s^r C_u^t q_4^{-1}$. By the previous paragraph and Lemma 2.8, there are positive elements $p_5$ and $q_5$ and nonnegative integers $m$ and $n$ such that $g_1 g_2 = p_5 C_n^m q_5^{-1}$. Since we can assume that $m < n+2$ by Lemma 5.6.v), $g_1 g_2 \in H$. Hence $H$ is a subgroup of $T_1$. Since $T_1$ is generated by $A = X_0$, $B = X_1$, and $C = C_1$, all of which are in $H$, $H = T_1$. □

THEOREM 5.8. *$T_1$ is simple.*

*Proof.* Suppose $N$ is a nontrivial normal subgroup of $T_1$, and let $\theta \colon T_1 \to T_1/N$ be the quotient homomorphism. Then there is an element $g \in T_1$ with $g \neq 1$ and $\theta(g) = 1$. By Theorem 5.7, $g = pC_n^m q^{-1}$ for some positive elements $p$, $q$ and nonnegative integers $m$, $n$ with $m < n+2$. Then

$$\theta(p^{-1}q) = \theta(C_n^m)$$

$$\text{and} \quad \theta\big((p^{-1}q)^{n+2}\big) = \theta\big((C_n^m)^{n+2}\big) = \theta\big((C_n^{n+2})^m\big) \overset{5.6.v)}{=} \theta(1) = 1\,.$$

By Lemma 5.4, there is a homomorphism $\alpha\colon F \to T_1/N$ defined on generators by $\alpha(A) = \theta(A)$ and $\alpha(B) = \theta(B)$. If $p^{-1}q \neq 1$, then $(p^{-1}q)^{n+2} \neq 1$, and so $\alpha(F)$ is a proper quotient group of $F$. Since every proper quotient group of $F$ is Abelian by Theorem 4.3, $\theta(AB) = \theta(BA)$. If $p^{-1}q = 1$, then $m, n > 0$ and $1 = \theta(C_n^m) \overset{5.6.iv)}{=} \theta(C_{n+1}^{m+1})\,\theta(X_{m-1}^{-1})$ and hence $\theta(X_{m-1}^{n+3}) = \theta\big((C_{n+1}^{m+1})^{n+3}\big) = \theta\big((C_{n+1}^{n+3})^{m+1}\big) = \theta(1) = 1$. It follows as before that $\theta(AB) = \theta(BA)$. Hence $\theta(A^{-1}BA) = \theta(B)$, so $\theta(A^{-1}C) = \theta(BA^{-2}C)$ by relation 4). Hence $\theta(BA^{-1}) = 1$, and so $\theta(B) = 1$ by relation 3). This implies that $\theta(A) = 1$. It now follows from relation 5) that $\theta(C) = 1$. Thus $N = T_1$, and so $T_1$ is simple. □

COROLLARY 5.9. *$T_1$ is isomorphic to $T$.*

## §6. THOMPSON'S GROUP $V$

As with the previous section, the material in this section is mainly from unpublished notes of Thompson [T1]; [T1] contains the statements of the lemmas (except for Lemma 6.2) and the statement and proof of Theorem 6.9, but does not contain the proofs of the lemmas.

Let $V$ be the group of right-continuous bijections of $S^1$ that map images of dyadic rational numbers to images of dyadic rational numbers, that are differentiable except at finitely many images of dyadic rational numbers, and such that, on each maximal interval on which the function is differentiable, the function is linear with derivative a power of 2. As before, it is easy to prove that $V$ is a group.

We can associate tree diagrams with elements of $V$ as we did for $F$ and $T$, except that now we need to label the leaves of the domain and range trees to indicate the correspondence between the leaves. For example, reduced tree diagrams for $A$, $B$, and $C$ are given in Figure 16.

Using the identification of $S^1$ as the quotient of $[0,1]$, define $\pi_0 : S^1 \to S^1$ by

$$\pi_0(x) = \begin{cases} \frac{x}{2} + \frac{1}{2}, & 0 \leq x < \frac{1}{2} \\ 2x - 1, & \frac{1}{2} \leq x < \frac{3}{4} \\ x, & \frac{3}{4} \leq x < 1. \end{cases}$$