# 5. The Gel'fand-Cetlin action

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To prove Proposition 4.5, it is enough to establish that for all $a \in \mathbf{C}P^1_r$, the tangent map $T_a\phi : T_a\mathbf{C}P^1_r \longrightarrow T_{\phi(a)}S^2_r$ satisfies

$$T_a\phi(Jv) = \widetilde{J}T_a\phi(v) \quad \text{and} \quad \widetilde{\omega}\big(T_a\phi(v), T_a\phi(Jv)\big) = 4\omega(v, Jv)\,.$$

By $U_2$-equivariance, we can restrict ourselves to $a = [\sqrt{r}, 0]$. The tangent space $T_a\mathbf{C}P^1_r$ is identified with $\{0\} \times \mathbf{C}$ and one can take $v = (0, 1)$ and $Jv = (0, i)$. One has $\phi(a) = (r, 0, 0)$,

$$T_a\phi(v) = (0, 2\sqrt{r}, 0)\,, \quad T_a\phi(Jv) = (0, 0, 2\sqrt{r}) = \widetilde{J}T_a\phi(v)$$

and $\widetilde{\omega}\big(T_a\phi(v), T_a\phi(Jv)\big) = 4$, while $\omega(v, Jv) = 1$. □

REMARKS

(4.6) The results of this section show that the spaces $\mathcal{P}^3_+(\alpha)$ for generic $\alpha$ are the symplectic leaves of the Poisson structure on the regular part of ${}^m\mathcal{P}^3_+$, or ${}^m\mathcal{P}\mathcal{P}^3_+$ given in (3.13) and (3.14).

(4.7) If one works in the pure quaternions $I\mathbf{H}$, the complex structure $\widetilde{J}$ on $S^2_r$ becomes

$$\widetilde{J}(v) = \frac{q\,v}{|q|}\,, \quad (v \in T_qS^2_r = I\mathbf{H})\,.$$

The sphere $S^2_r$ is a co-adjoint orbit of $U_1(\mathbf{H})$ and the Hermitian form $\widetilde{w}$ is the Kirillov–Kostant form (see [Gu, Theorem 1.1]).

(4.8) The isomorphism between the symplectic reductions of the Grassmannian $\mathbf{G}_2(\mathbf{C}^m)$ and the product of $\mathbf{C}P^1$'s that underlies our results 3.9, 4.4 and the proof of 4.5 is a symplectic version of the Gel′fand-MacPherson correspondence ([GM] and [GGMS]). The fact that this isomorphism comes from two reductions of $\mathcal{M}$ is the philosophy of "dual pairs" (see [Mo] and the references therein).

## 5. THE GEL′FAND-CETLIN ACTION

On ${}^m\mathcal{F}^k$ we have so far defined the length functions $\widetilde{\ell}$ measuring the distances between successive vertices. We now introduce $\widetilde{d} : {}^m\mathcal{F}^k \to \mathbf{R}^m$, $\widetilde{d}(\rho) = (|\rho(1)|, |\rho(1) + \rho(2)|, \dots, |\sum_{i=1}^m \rho(i)|)$, the lengths of the diagonals connecting the vertices to the origin. (Only $m-3$ of these functions are new, as $\widetilde{d}(\rho)_1 = \widetilde{\ell}(\rho)_1$, $\widetilde{d}(\rho)_{m-1} = \widetilde{\ell}(\rho)_m$, and $\widetilde{d}(\rho)_m = 0$. Hereafter we write only $\ell_i, d_i$ and the $\rho$ is to be understood.)

As with $\widetilde{\ell}$, the function $\widetilde{d}$ descends to continuous but only generically smooth functions $d$ on ${}^m\widetilde{\mathcal{P}}^k$, ${}^m\mathcal{P}^k_+$ and ${}^m\mathcal{P}^k$. It is smooth where no $d_i$ vanishes, that is to say the polygon does not return to the origin prematurely. We call such a polygon $P$ *prodigal* and call $\big(\ell(P), d(P)\big)$ a *prodigal value.* The set of prodigal polygons is open dense in ${}^m\mathcal{P}^k_+$ with complement of codimension $k$.

For $k = 3$, there is in [KM2] (see also [Kl], § 2.1) introduced an action of a torus $T^{m-3}$ on prodigal polygons; the $i$th circle acts by rotating the section of the polygon formed by the first $i$ edges about the $i$th diagonal. (When that diagonal is length zero, there is no well-defined axis about which to rotate, and indeed the action cannot be extended continuously over this subset.) This action plainly preserves the level sets of the functions $d$, but more is true:

THEOREM 5.1 (KM2). *On the subspace of prodigal polygons of $\mathcal{P}^3_+(\alpha)$, the function $d$ is a moment map for these "bending flows".*

One important consequence of this is that the torus action also preserves the symplectic structure. It does not, seemingly, preserve the Riemannian metric nor the complex structure (the codimension of the singular set is not even; see also § 6).

These functions $\ell, d$ lifted to $\mathbf{V}_2(\mathbf{C}^m)$ have simple matrix-theoretic interpretations. For $(a,b) \in \mathbf{V}_2(\mathbf{C}^m)$, $i = 1, \dots, m$, introduce the truncated matrices $M_i = \begin{pmatrix} a_1 & b_1 \\ \vdots & \vdots \\ a_i & b_i \end{pmatrix}$, the first $i$ rows of $(a,b)$. Then the $2 \times 2$ matrix

$$M_i^* M_i = \sum_{j=1}^{i} \begin{pmatrix} |a_j|^2 & \bar{a}_j b_j \\ a_j \bar{b}_j & |b_j|^2 \end{pmatrix}$$

has the eigenvalues

$$\frac{1}{2}\left( \sum_{j=1}^{i} (|a_j|^2 + |b_j|^2) \pm \sqrt{ \left( \sum_{j=1}^{i} (|a_j|^2 - |b_j|^2) \right)^2 + 4 \left| \sum_{j=1}^{i} a_j \bar{b}_j \right|^2 } \right).$$

These are calculable from $\ell$ and $d$, since

$$\ell\big(\Phi(a,b)\big) = \ell\big(\dots, \phi(a_i,b_i), \dots\big) = (\dots, |a_i|^2 + |b_i|^2, \dots)$$

and

$$d\big(\Phi(a,b)\big) = \Big(\dots, \Big|\sum_{j=1}^{i} \phi(a_j, b_j)\Big|, \dots\Big)$$
$$= \Big(\dots, \sqrt{\Big(\sum_{j=1}^{i}(|a_j|^2 - |b_j|^2)\Big)^2 + 4\Big|\sum_{j=1}^{i} a_j\bar{b}_j\Big|^2}, \dots\Big)$$

So $\sum_{j=1}^{i} \ell_j$ is the sum of the two eigenvalues of $M_i^* M_i$, whereas $d_i$ is the difference. (Note that $\ell_1 = d_1$ as promised; $M_1^* M_1$'s lesser eigenvalue is 0.)

This $(2 \times 2)$-matrix $M_i^* M_i$ has the same nonzero eigenvalues as the $i \times i$ matrix $M_i M_i^*$. The latter matrix is more relevant in that it is the upper left $i \times i$ submatrix of the $m \times m$ matrix $(a,b)\,(a,b)^*$ introduced in section (3.11).

This family of Hamiltonians — the eigenvalues of the upper left submatrices — has been studied already in [Th] and is called the classical Gel′fand-Cetlin system (our main reference is [GS1]). The linear relations established above between them and $d, \ell$ are summed up in the following

THEOREM 5.2. *The bending flows on ${}^m\mathcal{P}^3_+(\alpha)$ are the residual torus action from the Gel′fand-Cetlin system on the Grassmannian* $\mathbf{G}_2(\mathbf{C}^m)$.

The Gel′fand-Cetlin action on the flag manifold has always been rather mysterious (at least to us); it is pleasant that in this case it has a natural geometric interpretation.

The Gel′fand-Cetlin functions $\{e_{ij}\}_{j \le i}$ (the $j$th eigenvalue of the upper left $i \times i$ submatrix) satisfy some linear inequalities that can be established using the minimax description of eigenvalues [Fr, p. 149]:

$$e_{i,j} \le e_{i-1,j+1} \le e_{i,j+1}\,.$$

For the polygon space functions $l, d$ most of these say $0 \le 0$; for each $i = 0, \dots, n-1$ the nontrivial inequalities are

$$0 \le -d_i + \sum_{\iota=1}^{i} \ell_\iota \le -d_{i+1} + \sum_{\iota=1}^{i+1} \ell_\iota \le d_i + \sum_{\iota=1}^{i} \ell_\iota \le d_{i+1} + \sum_{\iota=1}^{i+1} \ell_\iota\,.$$

But these are transparent in our situation, as they are just the triangle inequalities!

$$(1)\qquad \begin{aligned} \ell_{i+1} &\le d_i + d_{i+1} \\ d_i &\le \ell_{i+1} + d_{i+1} \\ d_{i+1} &\le \ell_{i+1} + d_i \end{aligned}$$

(The first one, $d_i \leq \sum_{\iota=1}^{i} \ell_\iota$, can be proved inductively from the others starting from $d_0 = 0$.)

In [GS1] it is left as an exercise to show that (1) are the *only* inequalities satisfied; equivalently, that every point in the convex polytope $\Gamma_m \subset \mathbf{R}^m \times \mathbf{R}^m$ defined by them (and $d_0 = d_m = 0$ and $\sum_i \ell_i = 2$) is realized by some Hermitian matrix. We show this directly:

THEOREM 5.3. *The image of* ${}^m\mathcal{P}^{k\geq 2}$ *under the map* $(\ell, d)$ *is the whole polytope* $\Gamma_m$.

*Proof.* We construct the polygons directly, vertex by vertex — really establishing that each space ${}^m\widetilde{\mathcal{P}}^k(\alpha, \delta)$ is nonempty (and so its quotient by $SO(k)$ is as well). We must place each new vertex on the intersection of two $S^{k-1}$'s, one of radius $d_{i+1}$ from the origin, the other of radius $\ell_{i+1}$ from the previous vertex. The inequalities $\ell_{i+1} \leq d_i + d_{i+1}$ and $d_{i+1} \leq \ell_{i+1} + d_i$ rule out one $S^{k-1}$ containing the other; the third inequality $d_i \leq \ell_{i+1} + d_{i+1}$ rules out their being separated balls. So they intersect in an $S^{k-2}$, a point or the whole $S^{k-1}$, anywhere on which we may place the new vertex. $\square$

(5.4) REMARKS

1) While the map $\ell$ is equivariant with respect to the usual action of $S_m$ on $\mathbf{R}^m$, the map $d$ can only be made equivariant under the involution $[i \leftrightarrow (n-i)]$, and the polytope $\Gamma_m$ is correspondingly less symmetric than the hypersimplex $\Xi_m$.

2) That the image of $(\ell, d)$ is the same when restricted to planar polygons has the flavor of a more general theorem of Duistermaat [D] on restricting moment maps to the fixed-point sets of antisymplectic involutions. In fact Duistermaat's theorem does not apply directly, because the subset where $d$ is smooth (and a moment map) is noncompact; in any case we preferred to give a polygon-theoretic proof.

3) When $k = 3$ Theorem 5.1 guarantees that the bending torus acts simply transitively on the fiber over an interior point of $\Gamma_m$, making this fiber a torus $U(1)^{m-3}$ (or $O(1)^{m-3}$ when $k = 2$). Over a prodigal boundary point of $\Gamma_m$, the fiber is still a product of 0- or 1-spheres, but fewer of them.

4) Bending around other diagonals than the ones above can be done in the same way, the moment map lifted to $\mathbf{V}_2(\mathbf{C}^m)$ being the difference of the two eigenvalues of $M^*M$ for a corresponding submatrix $M$ of $(a, b) \in \mathbf{V}_2(\mathbf{C}^m)$. For instance, we take

$$M = \begin{pmatrix} a_2 & b_2 \\ a_3 & b_3 \\ a_4 & b_4 \end{pmatrix}$$

for the diagonal $\partial_{2,4} := \rho(2) + \rho(3) + \rho(4)$. The bending flows around two diagonals $\partial_{p,q}$ and $\partial_{p',q'}$ commute if and only if the pairs $\{p,q\}$ and $\{p',q'\}$ intersect or are unlinked in $\mathbf{R}/m\mathbf{Z}$.

## 6. Toric manifold structures on ${}^m\mathcal{P}^3_+(\alpha)$ for $m = 4, 5, 6$

In this section, we study examples of $\mathcal{P}^3_+(\alpha) \subset {}^m\mathcal{P}^3$ such that the $m-3$ diagonal functions $d_2, \dots, d_{m-2} : \mathcal{P}^3_+(\alpha) \longrightarrow \mathbf{R}$ never vanish. The whole space $\mathcal{P}^3_+(\alpha)$ consists of prodigal polygons and, by §5, the bending flows give an action of a big (i.e. half-dimensional) torus on $\mathcal{P}^3_+(\alpha)$. By Delzant's theorem (see [De], or [Gu, §1]), we can construct from the moment polytope $\Delta_\alpha$ alone a toric manifold which is equivariantly symplectomorphic to the space $\mathcal{P}^3_+(\alpha)$. This can be achieved also by [DJ,§1.5], though only up to equivariant diffeomorphism. The latter also gives the real part, the planar polygon space $\mathcal{P}^2(\alpha)$, as a $2^{m-3}$-sheeted branched cover of $\Delta_\alpha$. We sum up below some results of these constructions without writing all the details.

Without explicit mention of the contrary, $\alpha$ is supposed to be generic. Contrary to the previous sections, we do not require that the perimeter of our polygons is 2. It was necessary to fix the perimeter in order to define the map $\ell$ and the value 2 is the natural choice to deal with the map $\Phi : \mathbf{V}_2(\mathbf{C}^m) \longrightarrow {}^m\widetilde{\mathcal{P}}^k$. But ${}^m\mathcal{F}^k(\alpha)$ makes sense for any $\alpha \in \mathbf{R}^m_{\geq 0}$ and so do the various moduli spaces ${}^m\mathcal{P}^k(\alpha)$, etc. When $\sum \alpha_i = 2$, the polytope $\Delta_\alpha$ is a slice through the Gel'fand-Cetlin moment polytope $\Gamma_m$ of §5: for general $\alpha$ it is a homothetic copy of this section.

(6.1) $m = 4$: The condition which guarantees that $d_2$ never vanishes is $\alpha_1 \neq \alpha_2$ or $\alpha_3 \neq \alpha_4$. The space of quadrilaterals ${}^4\mathcal{P}^3_+(\alpha)$ is then a compact toric manifold of dimension 2, therefore diffeomorphic to $\mathbf{C}P^1$. The moment map $d_2$ has image the interval $\Delta_\alpha := I_1 \cap I_2$ where

$$I_1 := [|\alpha_1 - \alpha_2|, \alpha_1 + \alpha_2] \quad \text{and} \quad I_2 := [|\alpha_4 - \alpha_3|, \alpha_4 + \alpha_3].$$

The space ${}^4\mathcal{P}^2(\alpha)$ is $\mathbf{R}P^1$. The quadrilateral spaces ${}^4\mathcal{P}^2(\alpha)_+$ have long since been classified (see for instance [Ha]). One has

$${}^4\mathcal{P}^2(\alpha)_+ = \begin{cases} S^1 \sqcup S^1 & \text{when } I_1 \subset I_2 \text{ or } I_2 \subset I_1 \\ S^1 & \text{otherwise} \end{cases}.$$