# 3. Osculating indicatrices and Minkowski caustic

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## 3. OSCULATING INDICATRICES AND MINKOWSKI CAUSTIC

Consider the curve $J \subset \mathbf{R}^2$ centrally symmetric to the indicatrix $I$ with respect to the origin, and coorient it inwards. Since $I$ is the time-1 front of the origin, the time-1 map $\phi_1$ of the geodesic flow takes the foot points of all the cooriented contact elements of $J$ to the origin. If the curve $J$ is a source of light in our anisotropic Minkowski plane then light from all points of $J$ focuses at the origin in unit time.

Let $\gamma$ be a nonparametrized closed strictly convex curve in Minkowski plane, cooriented inwards. For every point $x \in \gamma$ there exists a unique curve $J(x)$, homothetic to $J$ (that is, obtained from $J$ by a dilation with a positive coefficient and a parallel translation) which is second order tangent to $\gamma$ at $x$.

DEFINITIONS. Call $J(x)$ the *osculating indicatrix* of $\gamma$ at $x$. The coefficient $r(x)$ of the dilation that takes $J$ to $J(x)$ is called the (Minkowski) *curvature radius* of $\gamma$ at $x$. The center of $J(x)$, i.e., the image of the origin under the homothety that takes $J$ to $J(x)$, is called the (Minkowski) *center of curvature* of $\gamma$ at $x$. A point $x \in \gamma$ is called a (Minkowski) *vertex* if the osculating indicatrix is third order tangent to $\gamma$ at $x$. Call the envelope $\Gamma$ of the Minkowski normals to $\gamma$ its (Minkowski) *caustic*.

REMARK. The curvature radius at $x \in \gamma$ is the focusing time for light, propagating from a small piece of $\gamma$ around $x$ in the direction of the coorientation. This time is positive if the coorientation vectors point to the convex side of the curve, and negative otherwise.

If the metric is Euclidean all these notions coincide with the usual ones, e.g., the osculating indicatrix is the osculating circle, etc. We list below a number of properties of osculating circles and Euclidean caustics subject to a generalization in the Minkowski setting.

1) The caustic of a curve is the locus of its centers of curvature.

2) A vertex of a curve corresponds to a singularity of its caustic.

3) A vertex is an extremum of the curvature radius.

4) The caustic of a generic curve is a piecewise smooth curve with an even number of cusps and without inflection points.

5) If a caustic is bounded then the alternating sum of the lengths of its smooth pieces equals zero.

6) A curve $\gamma$ is described by the free end of a stretched string developing from its caustic $\Gamma$.

7) (Kneser's theorem). The osculating circles of an arc of a curve, free from vertices, are pairwise disjoint and lie one inside the other.

In the case of Minkowski geometry these properties still make sense (using the above definitions) except for 5) and 6) which require an explanation because the Minkowski length of a curve depends on its orientation.

Give the normals of $\gamma$ the inward orientation; then every smooth piece of $\Gamma$ gets an orientation too. The length of a smooth oriented piece of $\Gamma$ is understood to be its length in Minkowski geometry. In this way property 5) makes sense — see Figure 1.
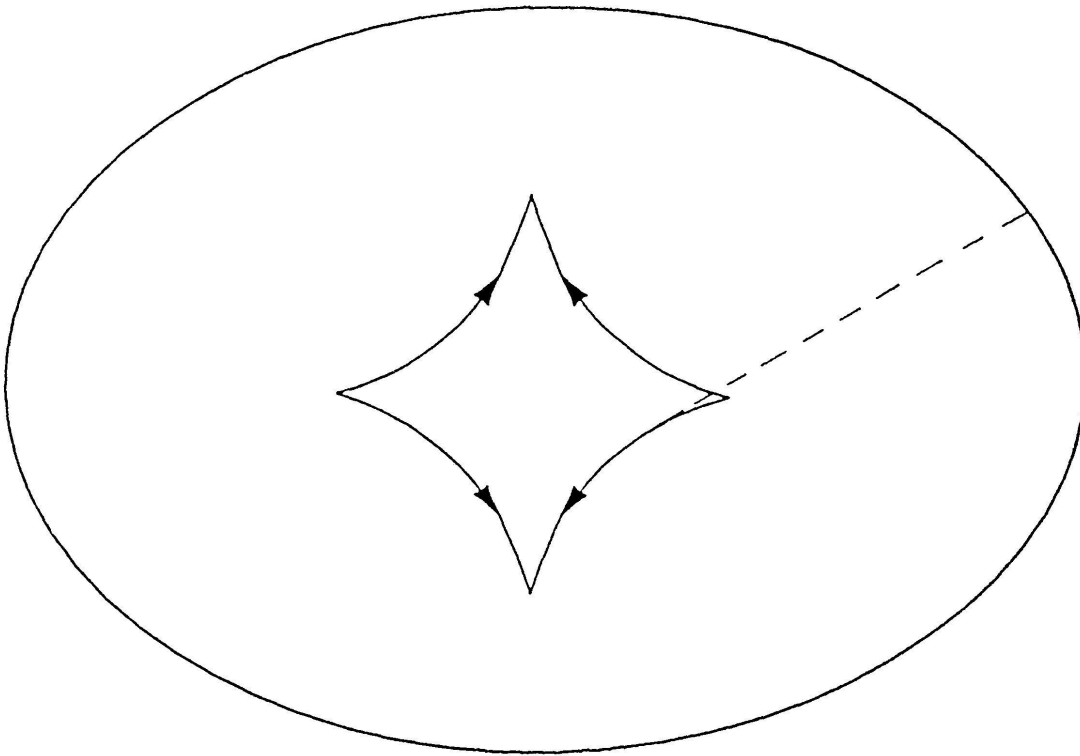


FIGURE 1

To explain property 6) consider a smooth arc of the caustic, oriented as above, and let $A$ and $B$ be two of its points such that $A$ precedes $B$ on the arc. Consider the tangent segments to $\Gamma$ at $A$ and $B$ which are normals to $\gamma$, oriented "from $\gamma$". Let $r$ and $R$ be their respective Minkowski lengths and $L$ be the Minkowski length of the arc $AB$ of the caustic. Property 6) asserts that $R - r = L$ — see Figure 2.
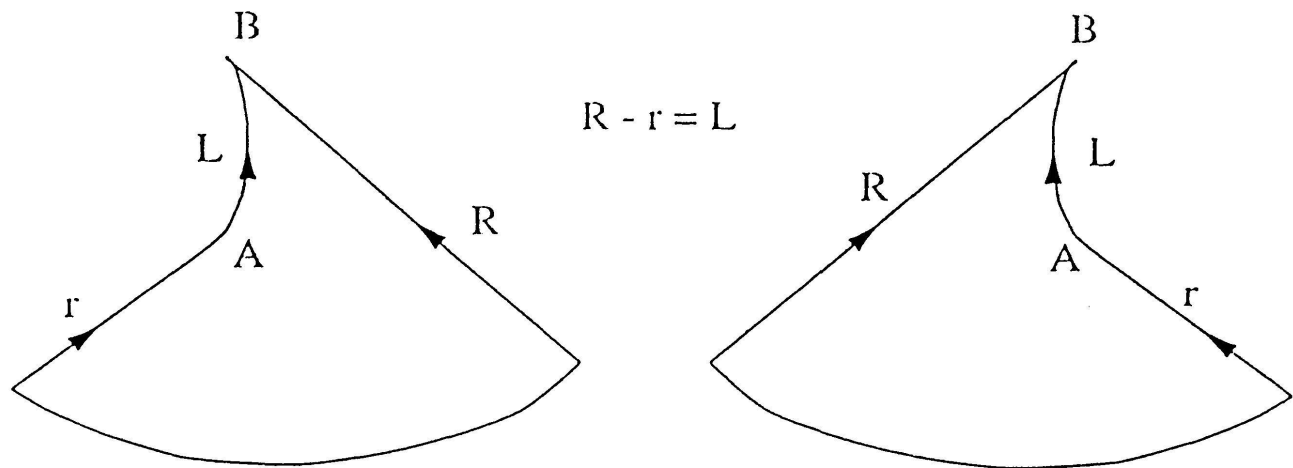
FIGURE 2

Various statements of the next theorem can be found (in an explicit or an implicit form) in the papers on plane Minkowski and relative geometry, mentioned in the References. I have not seen an approach to the proof via contact geometry in the literature.

THEOREM 3.1.  *The properties 1)–7) hold true in the Minkowski setting.*

*Proof.*  As before, $H$ denotes the Hamiltonian function associated with the Minkowski metric, $S = H^{-1}(1) \subset T^*\mathbf{R}^2$ and $\pi : S \to \mathbf{R}^2$ is the projection. Let $\widetilde{\gamma}$ be the lift of the cooriented curve $\gamma$ to $S$ (considered as the space of cooriented contact elements of the plane), and let $Z \subset S$ be the cylinder that consists of the trajectories of the Hamiltonian vector field $\xi$ through $\widetilde{\gamma}$. Denote by $\widetilde{\Gamma} \subset Z$ the curve consisting of points at which the rank of the projection $\pi|_Z$ is less than 2. Thus $\widetilde{\Gamma}$ is the set of points at which the fibers of $\pi$ are tangent to $Z$. Since the trajectories of $\xi$ project diffeomorphically to the plane the rank of $\pi|_Z$ equals 1 along $\widetilde{\Gamma}$. The curve $\widetilde{\Gamma}$ projects to the caustic $\Gamma$.

To prove property 1), consider the osculating indicatrix $J(x)$ at $x \in \gamma$, cooriented inwards. Then $\widetilde{J}(x) \subset S$ is tangent to $\widetilde{\gamma}$ at point $\widetilde{x}$, the cooriented contact element of $\gamma$ at $x$. Let $r(x)$ be the curvature radius of $\gamma$ at $x$. Then $\phi_{r(x)}\big(\widetilde{J}(x)\big)$ is a fiber of $\pi$. Therefore a fiber of $\pi$ is tangent to the curve $\phi_{r(x)}(\widetilde{\gamma}) \subset Z$, and hence $\phi_{r(x)}(\widetilde{x}) \in \widetilde{\Gamma}$. It remains to note that $\pi\big(\phi_{r(x)}(\widetilde{x})\big)$ is the center of curvature of $\gamma$ at $x$.

Likewise, if $x \in \gamma$ is a vertex then $\widetilde{\Gamma}$ is tangent to the curve $\phi_{r(x)}\big(\widetilde{J}(x)\big)$ at point $\phi_{r(x)}(\widetilde{x})$. Therefore $\widetilde{\Gamma}$ is tangent to a fiber of $\pi$, so $\Gamma$ has a singularity at the respective center of curvature. Property 2) follows. It follows also that the singularities of the caustic are the singularities of the projection $\pi : \widetilde{\Gamma} \to \Gamma$; the curve $\widetilde{\Gamma}$ is smooth.

Next, note that an orientation of $\gamma$ gives $\Gamma$ a coorientation. Give $\Gamma$ an orientation; then the pair (orienting vector, coorienting vector) is either a positive or a negative frame along each smooth piece of $\Gamma$. The positive and negative pieces alternate, so the number of cusps is even.

Consider the space of oriented lines in the plane (topologically, the cylinder); the tangent lines to the caustic constitute a curve $\sigma$ in this space. The family of Minkowski normals to $\gamma$ being smooth, the curve $\sigma$ is smooth as well. An inflection of $\Gamma$ would correspond to a singularity of $\sigma$. Thus $\Gamma$ is inflection free, and property 4) follows. Note that an inflection of $\Gamma$ corresponds to the tangency between $\widetilde{\Gamma}$ and a trajectory of the field $\xi$. Therefore $\widetilde{\Gamma}$ is transverse to $\xi$.

Vertices correspond to the stationary osculating circles, therefore they are extrema of the curvature radius. Conversely, consider a critical value of the curvature radius at $x \in \gamma$, and assume that the caustic is smooth at the corresponding curvature center. Then the direction of $\Gamma$ is parallel to the tangent line to $\gamma$ at $x$. However the tangent line to $\Gamma$ is the Minkowski normal to $\gamma$ at $x$ which is transverse to $\gamma$. Property 3) follows.

One may use the Minkowski length of the tangent segment to $\Gamma$ from $\gamma$ that is, the curvature radius $r$, as a local parameter on a smooth oriented piece of the caustic. The velocity vector $\partial\Gamma/\partial r$ at a point of $\Gamma$ is the projection under $d\pi$ of the vector $\xi$ at the corresponding point of $\widetilde{\Gamma}$. Therefore the vector $\partial\Gamma/\partial r$ belongs to the indicatrix, and the parameterization $\Gamma(r)$ is by arc-length. Property 6) follows. Property 5) is obtained from 6 by summation over smooth pieces of the caustic.

Equivalently, the argument from the preceding paragraph means that the Minkowski length of a smooth arc $\delta$ of the caustic, oriented as above, equals the integral of the contact form $\lambda$ over the lifted arc $\widetilde{\delta} \subset \widetilde{\Gamma}$. Likewise, $r$ and $R$ are the respective integrals of $\lambda$ over the trajectory segments of the field $\xi$.

Since $i_\xi d\lambda = 0$, the integral of $d\lambda$ over the quadrilaterals in $Z$, bounded by the trajectories of $\xi$ and the curves $\widetilde{\delta}$ and $\widetilde{\gamma}$, vanishes. Applying Stokes' theorem and taking into account that $\lambda = 0$ on $\widetilde{\gamma}$, the equality $L - R + r = 0$ follows.

To prove property 7), the Kneser theorem, assume that two osculating indicatrices intersect at some point $C$. Let $A$ and $B$ be the respective centers of curvature such that $A$ precedes $B$ on the oriented smooth piece of the caustic, and let $r$ and $R$ be the corresponding curvature radii. Then the length of the oriented segments $CA$ and $CB$ equal $r$ and $R$, respectively. By property 6) the Minkowski length of the arc $AB$ equals $R - r$, and this violates the triangle inequality — see Figure 3.
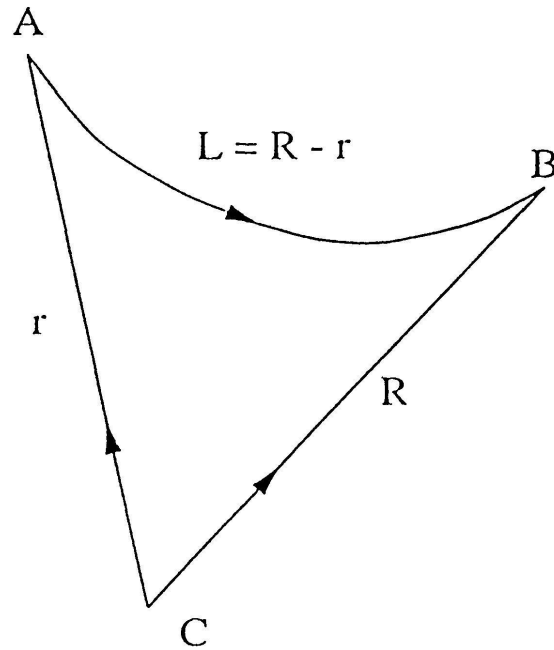
FIGURE 3

REMARK. The definitions given at the beginning of this section extend to complete Finsler metrics without conjugate points. Properties 1) – 7) hold in this case as well, and the proof goes through without change.

Returning to the situation of the Introduction one sees that Theorem 0.1 is the 4-vertex theorem in the Minkowski geometry associated with a parametrized curve (as explained in Section 2). In particular, the envelope $\Gamma(t)$ of the lines $l(t)$ is the Minkowski caustic. We collect explicit formulas in the next lemma. These formulas hold true even if the function $\left[\gamma''(t), \gamma'''(t)\right]$ has zeroes.

LEMMA 3.2.   *The envelope is*

$$\Gamma(t) = \gamma(t) + \frac{\left[\gamma'(t), \gamma''(t)\right]}{\left[\gamma''(t), \gamma'''(t)\right]}\ \gamma''(t)\,,$$

*the radius of curvature is*

$$r(t) = \frac{\left[\gamma'(t), \gamma''(t)\right]^2}{\left[\gamma''(t), \gamma'''(t)\right]}\,,$$

*and cusps of $\Gamma(t)$ correspond to critical points of the curvature function*

$$k(t) = \frac{\left[\gamma''(t), \gamma'''(t)\right]}{\left[\gamma'(t), \gamma''(t)\right]^2}\,.$$

EXAMPLES.

1) Let $\gamma$ be a nonparametrized smooth closed strictly convex plane curve and $O$ be its interior point. Take $O$ as the origin in $\mathbf{R}^2$. There exists a parameterization $\gamma(t)$ such that $[\gamma(t), \gamma'(t)] = 1$ for all $t$. Then $\gamma''(t)$ is colinear with $\gamma(t)$, and the caustic in the corresponding Minkowski geometry degenerates to the point $O$. All points of $\gamma$ are Minkowski vertices, and all osculating indicatrices coincide with the curve itself.

2) Let a parameterization $\gamma(t)$ satisfy $[\gamma'(t), \gamma''(t)] = 1$ for all $t$ (an affine parameter). The indicatrix in the corresponding Minkowski geometry is given by the formula $I(t) = \gamma''(t)$. The lines $l(t)$, generated by the vectors $\gamma''(t)$, are called affine normals of the curve. The line $l(t)$ is tangent to the curve that consists of midpoints of the segments, bounded by the intersections of $\gamma$ with the lines, parallel to the tangent line to $\gamma$ at point $\gamma(t)$ — see Figure 4. The envelope of the affine normals is called the affine caustic.
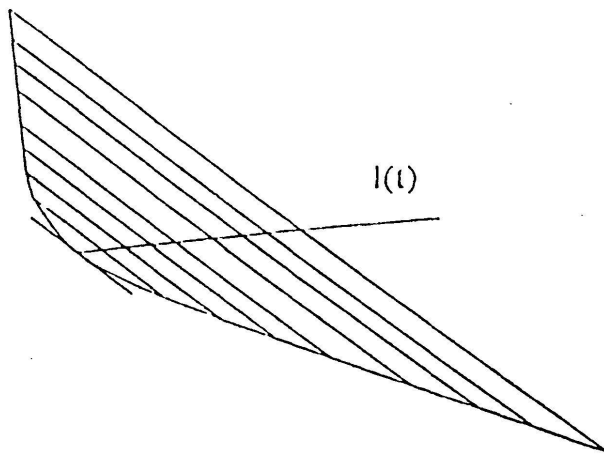


FIGURE 4

Differentiating the equality $[\gamma'(t), \gamma''(t)] = 1$ one finds: $\gamma'''(t) = -k(t) \gamma'(t)$, where the function $k(t)$ is called the affine curvature. The affine curvature is reciprocal to the curvature radius in the corresponding Minkowski geometry. Critical points of the affine curvature are called affine vertices (or sextactic points). A smooth closed convex curve has at least 6 affine vertices (see [Bl 2]); thus a generic affine caustic has at least 6 cusps. Affine vertices are points of 5-th order contact of the curve with a conic; at an ordinary point the order of contact is one less.

To conclude this section, note that the Minkowski metric gives rise to a symplectic form $\omega$ in the space $C$ of oriented lines in the plane. Indeed, $C$ is

identified with the space of trajectories of the geodesic flow $\xi$. Let $\lambda$ be the contact form in the space of cooriented contact elements associated with the Hamiltonian function $H$ (see Theorem 1.1). Then the 2-form $d\lambda$ descends to $C$; this is the symplectic form in question.

The family of Minkowski normals to $\gamma$ is a curve $\sigma \subset C$. Let $\sigma_0 \subset C$ be the curve that consists of oriented lines through a fixed point $x$ in the plane.

LEMMA 3.3. *The $\omega$-area of the region in $C$ between the curves $\sigma$ and $\sigma_0$ equals zero.*

*Proof.* Denote by $\widetilde{\gamma}_0$ the set of cooriented contact elements with the foot point at $x$. Then $\widetilde{\gamma}_0$ is a Legendrian curve. The projections of $\widetilde{\gamma}$ and $\widetilde{\gamma}_0$ along the trajectories of $\xi$ are the curves $\sigma$ and $\sigma_0$. The area under consideration is the integral of the form $d\lambda$ over a film spanned by $\widetilde{\gamma}$ and $\widetilde{\gamma}_0$. By Stokes' theorem, this area equals

$$\int_{\widetilde{\gamma}} \lambda - \int_{\widetilde{\gamma}_0} \lambda = 0$$

since both curves are Legendrian.

In particular, the curves $\sigma$ and $\sigma_0$ intersect at least twice. Therefore at least two Minkowski normals to $\gamma$ pass through an arbitrary point $x$ in the plane. If the Minkowski metric is associated with a parametrized curve $\gamma(t)$ then the corresponding values of $t$ are the critical points of the function $\left[ \gamma(t) - x, \gamma'(t) \right]$.

REMARK. In the Euclidean case a convex closed curve has at least 2 double normals (chords, perpendicular to the curve at both ends). This is still true in the Minkowski setting, provided the indicatrix is centrally symmetric, but does not seem to hold in general.

## 4. MINKOWSKI VERTICES AND CHEBYSHEV SYSTEMS

This section contains proofs of the 4-vertex theorem in the Minkowski setting (different from the one in [T1]) and a generalization of Theorem 0.1. The arguments used are, more or less, classical; recently the approach via Sturm theory attracted new interest (see [A 1, A 4, A 5, G-M-O]).

Let $J$ have the same meaning as in the previous section and let $J(t)$ be some parameterization of this curve, $0 \le t \le T$. Let $\gamma(t)$ be a strictly convex closed smooth curve, parametrized so that the tangent vector $\gamma'(t)$ has the