

## 5. A WORKED EXAMPLE

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Important work of Gabai and Kazez [5], [6] which uses three-dimensional topology shows that, if  $\alpha_1, \alpha_2: G_1 \rightarrow G_2$  are nonzero-degree homomorphisms between infinite surface groups, they are strongly equivalent if and only if  $\mathcal{G}(\alpha_1) = \mathcal{G}(\alpha_2)$ ,  $\alpha_1(G_1) = \alpha_2(G_1)$  and  $\alpha_1(G_1^+) = \alpha_2(G_1^+)$ . They also show that, if  $\alpha_1, \alpha_2: G_1 \rightarrow G_2$  are homomorphisms between surface groups at least one of which is finite, then  $\alpha_1, \alpha_2$  are strongly equivalent if and only if  $\mathcal{G}(\alpha_1) = \mathcal{G}(\alpha_2)$ ,  $\alpha_1(G_1) = \alpha_2(G_1)$  and  $\alpha_1(G_1^+) = \alpha_2(G_1^+)$ .

## 5. A WORKED EXAMPLE

In this section we will apply the algorithm to a rather trivial example to illustrate the algebraic manipulations involved.

Consider the homomorphism  $\alpha: \langle a, b, c, d \mid (a, b)(c, d) \rangle \rightarrow \langle x, y \mid (x, y) \rangle$  induced by the homomorphism of free groups  $A: \langle a, b, c, d \mid \rangle \rightarrow \langle x, y \mid \rangle$  determined by  $(a, b, c, d) \mapsto (x, y, x, y^{-1})$ .

We have

$$\begin{aligned} A((a, b)(c, d)) &= (x, y)(x, y^{-1}) = (x, y)x^{-1}yx(x, y)^{-1}x^{-1}y^{-1}x \\ &= (x, y)^{1-x^{-1}y^{-1}x}. \end{aligned}$$

Since  $\alpha$  is orientation-true, Kneser's Theorem 4.8 implies that  $\mathcal{G}(\alpha)$  is obtained by applying the orientation map to  $1 - x^{-1}y^{-1}x$ , so  $\mathcal{G}(\alpha) = 0$ . Thus we want to apply the algorithm to transform  $A$  into a map  $A'$  inducing  $\alpha$ , such that  $A'((a, b)(c, d)) = 1$ .

Form the CW-surfaces associated with the given surface group presentations, so the free group generators can be viewed as loops.

Let us subdivide  $y$  into two edges, one again called  $y$ , and the other called  $z$ . We will call the vertices  $u$  and  $v$ , so that  $x$  is a loop at  $v$ ,  $y$  joins  $v$  to  $u$ , and  $z$  joins  $u$  to  $v$ . The algorithm requires us to subdivide  $x$ , but, in order to keep the example simple, we shall not do this. Now we subdivide  $b$  and  $d$  into two edges labelled  $y1, z1$ , and  $y2, z2$  respectively. Here the first letter indicates the image label, while, since we plan to depict the moves in planar diagrams, we also want a label to identify equal edges, and it is convenient to use integers for this identification. Similarly, we label  $c$  and  $d$  as  $x1$  and  $x2$ , respectively.

We first use Construction 2.9 to get a cellular map, and hence a diagram, and then, after some simple applications of Construction 3.5 and 3.11, we can obtain the first diagram in Figure 5.1. Now we can apply the two-step

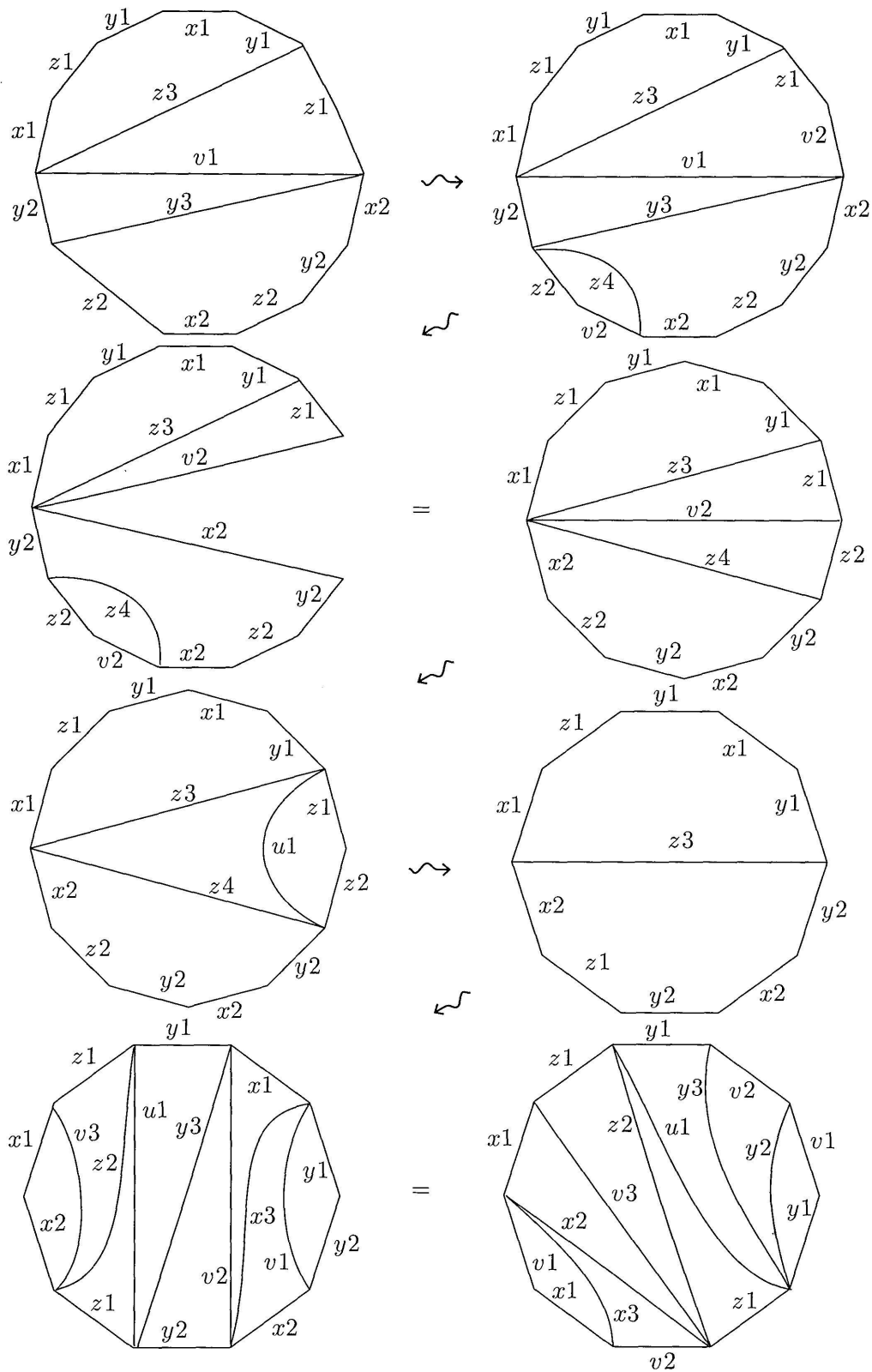


FIGURE 5.1

A worked example

Construction 3.18, to pass from the first to the second, and the second to the third diagram. Thus the first and third diagrams are obtained from the second, by first collapsing  $v_1$  and  $v_2$ , respectively, and then identifying  $z_2 = z_4$ , and  $y_2 = y_3$ , respectively. The fourth diagram is a convenient redrawing of the

third diagram. Now we apply Construction 3.16 (a) to identify  $z_1 = z_2$  and  $z_3 = z_4$ , and arrive at the sixth diagram in Figure 5.1.

Now we can apply Construction 3.24 to arrive at the seventh diagram in Figure 5.1, where we have three punctured spheres, as depicted in Figure 5.2, and we see that  $v_1, v_2$  are non-separating non-trivial  $V$ -loops, and cutting along these leaves a sphere with four punctures, which can be opened up into a disc by cutting along  $x_1, y_1$ , and  $z_1$ . Thus we can rearrange the seventh diagram in Figure 5.1 to obtain the eighth diagram.

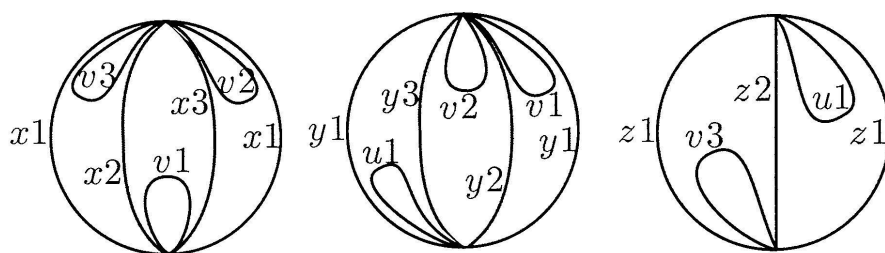


FIGURE 5.2  
A normal form

To see what this says about our original group homomorphism, we express all the steps algebraically, by manipulating groupoid presentations.

Here  $X_2$  has only one face, and we have to choose a maximal subtree, and it is natural to choose  $\{u, v, z\}$ . Let us express this by writing  $\langle x, y; z \mid (x, yz) \rangle$ , where the edges after the semicolon specify (the edge set of) a maximal subtree among the boundary edges. Recall that for CW-surface fundamental groupoid presentations we do not specify vertices, since they correspond to face-adjacency cycles.

In the same spirit, we express the first diagram in Figure 5.1 as

$$\langle x_1, y_1, x_2, y_2; z_1, z_2; v_1, y_3, z_3 \mid \overline{x_1} \overline{z_1} \overline{y_1} \overline{x_1} y_1 z_3, \overline{z_3} z_1 v_1, \overline{v_1} y_3 y_2, \overline{y_3} \overline{x_2} y_2 z_2 x_2 \overline{z_2} \rangle,$$

where the edges after the second semicolon specify the edges to be erased to form a single face, and overlines indicate inverses. Here we can identify  $a = x_1, b = y_1 z_1, c = x_2, d = \overline{z_2} \overline{y_2}$ .

Now we introduce a new vertex, two new edges  $v_2, z_4$ , and a new face  $z_4 = z_2 v_2$ , so the second diagram is expressed as

$$\langle x_1, y_1, x_2, y_2; z_1, z_2, v_2; v_1, y_3, z_3, z_4 \mid \overline{x_1} \overline{z_1} \overline{y_1} \overline{x_1} y_1 z_3, \overline{z_3} z_1 v_2 v_1, \overline{v_1} y_3 y_2, z_4 \overline{v_2} \overline{z_2}, \overline{y_3} \overline{x_2} y_2 z_2 x_2 \overline{z_2} \rangle,$$

and here  $x_2$  is an arc, and we identify  $a = x_1$ ,  $b = y_1z_1$ ,  $c = x_2\overline{v_2}$ ,  $d = \overline{z_2}y_2$ . (We can recover the first diagram by collapsing  $v_2$  in the maximal subtree, and cashing in the new face relation to identify  $z_4 = z_2$ .)

We now collapse the edge  $v_1$  in the maximal subtree, and cash in the face relation  $(v_1)y_2 = y_3$ , to identify the two edges  $y_2 = y_3$ . This obliges us to choose new edges to erase, and we find that the fourth, and third, diagrams are expressed as

$$\langle x_1, y_1, x_2, y_2; z_1, z_2; z_3, z_4, v_2 \mid \overline{x_1} \overline{z_1} \overline{y_1} x_1 y_1 z_3, \overline{z_3} z_1 v_2, z_4 \overline{v_2} \overline{z_2}, \overline{y_2} \overline{x_2} y_2 z_2 x_2 \overline{z_4} \rangle,$$

and here we identify  $a = x_1$ ,  $b = y_1z_1$ ,  $c = x_2\overline{v_2} = \overline{z_2}y_2x_2y_2z_2$ ,  $d = \overline{z_2}y_2$ .

We now re-triangulate, and the fifth diagram can be expressed as

$$\langle x_1, y_1, x_2, y_2; z_1, z_2; z_3, z_4, u_1 \mid \overline{x_1} \overline{z_1} \overline{y_1} x_1 y_1 z_3, \overline{z_3} \overline{u_1} z_4, u_1 z_1 \overline{z_2}, \overline{y_2} \overline{x_2} y_2 z_2 x_2 \overline{z_4} \rangle,$$

and here we identify  $a = x_1$ ,  $b = y_1z_1$ ,  $c = \overline{z_2}y_2x_2y_2z_2$ ,  $d = \overline{z_2}y_2$ .

We now collapse the edge  $u_1$ , and make identifications using the face relations  $z_2 = (u_1)z_1$ ,  $z_4 = (u_1)z_3$ , and the sixth diagram can be expressed as  $\langle x_1, y_1, x_2, y_2; z_1; z_3 \mid \overline{x_1} \overline{z_1} \overline{y_1} x_1 y_1 z_3, \overline{y_2} \overline{x_2} y_2 z_1 x_2 \overline{z_3} \rangle$ , and here we identify  $a = x_1$ ,  $b = y_1z_1$ ,  $c = \overline{z_1}y_2x_2y_2z_1$ ,  $d = \overline{z_1}y_2$ .

We now retriangulate, to express relations which map to relations in the free group.

Notice that we have now lifted  $\alpha$  to the homomorphism

$$A': \langle a, b, c, d \mid \rangle \rightarrow \langle x, y \mid \rangle$$

determined by  $(a, b, c, d) \mapsto (x, y, \overline{y}xy, \overline{y})$ , and  $A'((a, b)(c, d)) = 1$ .

Moreover, by changing presentations, we can now express  $\alpha$  in a more natural form. We take the non-separating  $v$ -loops  $v_1 = y_1\overline{y_2}$  and  $v_2 = x_1y_1\overline{y_2}x_2$ , and get the presentation

$$\langle x_1, y_1, v_1, v_2; z_1 \mid \overline{x_1} \overline{z_1}, \overline{y_1} v_2 \overline{v_1} y_1 z_1 \overline{v_2} x_1 v_1 \rangle,$$

and here we identify  $x_2 = \overline{v_2}x_1v_1$ ,  $y_2 = \overline{v_1}y_1$ , so  $a = x_1$ ,  $b = y_1z_1$ ,  $c = \overline{z_1}y_2x_2y_2z_1 = \overline{z_1}y_1v_1\overline{v_2}x_1y_1z_1$ ,  $d = \overline{z_1}y_2 = \overline{z_1}y_1v_1$ .

Now we can collapse the maximal subtrees, and we have a description of our group homomorphism as follows. We have the genus two surface group  $\langle x_1, y_1, v_1, v_2 \mid \overline{x_1} \overline{y_1} v_2 \overline{v_1} y_1 \overline{v_2} x_1 v_1 \rangle$ , we first impose relations annihilating the two generators  $v_1, v_2$ , to get a free group, and we then impose a relation to get the genus one surface group. Here we can identify  $a = x_1$ ,  $b = y_1$ ,  $c = \overline{y_1} v_1 \overline{v_2} x_1 y_1$ ,  $d = \overline{y_1} v_1$ , and thus  $bd = v_1$ ,  $ab\overline{c}d = v_2$ . This represents  $\alpha$  in one of the normal forms described in Case 4.2.

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