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QUATERNARY CUBIC FORMS AND PROJECTIVE ALGEBRAIC THREEFOLDS

by Alexander SCHMITT

INTRODUCTION

As algebraic geometers, we are interested in a special kind of complex manifolds, namely in complex submanifolds of projective spaces. A submanifold X of $\mathbf{P}_n$ is given as the common zero locus of a set of homogeneous polynomials such that the Jacobi matrix of these polynomials has rank $n - \dim X$ at every point of X. We call such a manifold a *projective algebraic manifold*. The main goal is the classification of projective algebraic manifolds up to biholomorphic equivalence. Now, a projective algebraic manifold is in particular an oriented and closed topological manifold. Moreover, biholomorphic maps are orientation preserving homeomorphisms.

Thus, we obtain a natural approach to the classification of projective algebraic manifolds which can be stated for complex dimension 3 as follows:

Given a six-dimensional, closed, and oriented topological manifold X, describe all projective algebraic threefolds (up to biholomorphic equivalence) whose underlying topological manifold is orientation preservingly homeomorphic to X.

Of course, one does not have a general classification of the respective topological manifolds. However, if we restrict our attention to simply connected, six-dimensional, closed, and oriented topological manifolds with torsion free homology, there is a classification result in the sense of algebraic topology, due to C.T.C. Wall [Wa] and P.E. Jupp [Ju]. This means the classification of simply connected, six-dimensional, closed, and oriented topological manifolds with torsion free homology up to orientation preserving homeomorphy can be reduced to the classification of certain algebraic data, so called admissible systems of invariants.

The explicit classification of these algebraic data can be carried out in the case the second Betti number b_2 is 1 [OV]. But already for $b_2 = 2$, the picture is rather complicated and not yet complete [Sch3]. So, it seems to be a rather hopeless task to classify systems of invariants for $b_2 > 2$. Thus, we restrict ourselves to the consideration of the most important part of the system of invariants of the simply connected, six-dimensional, closed, and oriented topological manifold X, the cup form

$$\begin{array}{rccc} \varphi_X: & S^3H^2(X,\mathbf{Z}) & \longrightarrow & \mathbf{Z} \\ & [a\otimes b\otimes c] & \longmapsto & (a\cup b\cup c)[X]\,. \end{array}$$

Here, $[X]$ is the fundamental class of X. We remark that the assumptions we make on the manifold X imply that the whole cohomology ring of X is determined by φ_X and the third Betti number $b_3(X)$.

We can also replace $\mathbf{Z}$ by $\mathbf{R}$ or $\mathbf{C}$ to obtain a weaker invariant. By our hypothesis, $H^2(X,\mathbf{Z})$ is a free $\mathbf{Z}$-module, and $H^2(X,R) = H^2(X,\mathbf{Z})\otimes_{\mathbf{Z}} R$, $R = \mathbf{R},\mathbf{C}$. If we fix a basis for $H^2(X,R)$, we can identify φ_X with a homogeneous cubic polynomial. On the module of all homogeneous cubic polynomials in b variables, there is an action of $\mathrm{GL}_b(R)$ by substitution of variables. Hence, we obtain a coarse picture of the classification of simply connected, six-dimensional, closed, and oriented topological manifolds with $b_2 = b$ if we determine the normal forms for cubic polynomials over $\mathbf{Z}$ in b variables w. r. t. the action of $\mathrm{GL}_b(\mathbf{Z})$ and if we describe the set of forms φ_X, X being a topological manifold.

For the latter part, we remark that there is a simple criterion to check whether a given cubic polynomial over $\mathbf{Z}$ is of the form φ_X or not (see [Sch2], Cor. 1). For example, this criterion is fulfilled if all coefficients are divisible by 6. The determination of normal forms is again very difficult. However, if we work over the field of complex numbers instead, results are known for up to $b = 4$ variables. The results for $b \leq 3$ variables are easily accessible. On the other hand, the results for $b = 4$ are scattered in the literature of over 100 years. Hence, we have written an extensive summary of the theory of complex quaternary cubic forms. Being interested in (Cubic forms over $\mathbf{Z})/\mathrm{GL}_b(\mathbf{Z})$, it is more reasonable to consider the action of $\widetilde{\mathrm{SL}}_b(\mathbf{C}) := \{m \in \mathrm{GL}_b(\mathbf{C}) \mid \det(m) = \pm 1\}$. To simplify things we will consider the action of $\mathrm{SL}_b(\mathbf{C})$ instead. This is the content of Part I.

In the second part, we treat the following weakened form of our original problem:

Which quaternary cubic forms can occur as cup forms of simply connected projective threefolds?

For the case $b \leq 3$, we refer the reader to [OV]. In this part, we have collected a number of examples. We also show that there is a simply connected projective threefold with $b_2 = 3$ whose cup form defines a plane cubic with a node, a problem which remained unsolved in [OV]. We conclude our notes by a brief summary of the author's results concerning the non-realizability of certain *real* cubic polynomials as cup forms of projective threefolds.

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I. QUATERNARY CUBIC FORMS

In this section, we will be concerned with the space $S^3(\mathbf{C}^{4^\vee})$ of quaternary cubic forms on which $\mathrm{SL}_4(\mathbf{C})$ acts by substitution of variables. In particular, we will treat the following problems:

1) Find "good" representatives for the orbits in $S^3(\mathbf{C}^{4^\vee})$;

2) Describe the categorical quotient $S^3(\mathbf{C}^{4^\vee}) /\!/ \mathrm{SL}_4(\mathbf{C})$.

(The categorical quotient is an affine algebraic variety whose set of points is in natural bijection with the closed orbits in $S^3(\mathbf{C}^{4^\vee})$. A good introduction to this kind of constructions can be found in [Ne].)

1. Normal Forms for Quaternary Cubic Forms

1.1. *Normal Forms for Quaternary Cubic Forms Defining Non-Singular Cubic Surfaces.* Here, the result is as follows:

THEOREM 1. *Every homogeneous polynomial of degree* 3 *in four variables defining a non-singular cubic surface can be brought into one of the following canonical forms* ($r_i, r, s, t \in \mathbf{C}^*$)*:*

$$(*) \qquad r_1x_1^3 + r_2x_2^3 + r_3x_3^3 + r_4x_4^3 + r_5(-x_1 - x_2 - x_3 - x_4)^3,$$

$$\text{where } \sum_{i=1}^{5} \pm 1/\sqrt{r_i} \neq 0 \qquad \textit{(Sylvester's pentahedral form)}$$

$$(*_1) \qquad r(x_1^3 + x_2^3 + x_3^3 + x_4^3) \qquad \textit{(diagonal form)}$$

$$(*_2) \qquad rx_1^3 + x_2^3 + x_3^3 + x_4^3 - 3sx_2x_3x_4,$$

$$\text{where } (s^3 - 1)(s^3 + 8) \neq 0 \qquad \textit{(non-equianharmonic form)}$$

$$(*_3) \qquad x_2^3 + x_3^3 + x_4^3 - 3x_1^2(r_2x_2 + r_3x_3 + r_4x_4)$$

$$(*_4) \qquad x_2^3 + x_3^3 + x_4^3 - 3x_1^2(r_1x_1 + r_2x_2 + r_3x_3 + r_4x_4)$$

$$(*_5) \qquad 2rx_1^3 + x_2^3 + x_3^3 - 3x_1(sx_1x_2 + x_1x_3 + tx_4^2),$$

$$\text{where } st(r \pm s^{\frac{3}{2}} \pm 1) \neq 0.$$

For a proof of this theorem, we refer the reader to Segre's book [Se]. We will also call a form being equivalent to a form of type $(*)$ a *Sylvestrian pentahedral form.* Such a form determines a configuration of five planes which is called the *Sylvester pentahedron.* Forms being equivalent to diagonal or non-equianharmonic forms will be called *degenerate Sylvestrian pentahedral forms.*

REMARK 1. Given a cubic form f defining a non-singular cubic surface, one is led to ask to which of the above forms f is equivalent. This problem is related to the geometry of the Hessian surface $H_f = 0$ in the following way:

If the Hessian surface is reducible, there are two possibilities: Either it consists of four different planes or of a cone over a smooth plane cubic and a plane. In the first case, f is equivalent to a diagonal form, and in the second case, f is equivalent to a non-equianharmonic form.

If the Hessian surface is irreducible, we have to look at its singularities. If there are precisely ten A_1-singularities, f is equivalent to a Sylvestrian pentahedral form, and the Sylvester pentahedron is determined by the configuration of the singular points. If there are seven singular points, one A_1-singularity and six A_k-singularities with $k \geq 2$, then f is equivalent to a form $(*_3)$ or $(*_4)$ depending on whether the intersection of the Hessian surface with the tangent cone to the A_1-singularity consists of a double line and an irreducible conic or of a double line and a reducible conic. If there are four singular points on the Hessian surface, then f can be brought into a form of type $(*_5)$. In any case, much information on the canonical form can be read off the configuration of the singular points of $H_f = 0$. We refer the reader to [Se] and [Sch1] for the details.

The following results on canonical forms of quaternary cubic forms can be easily derived from the treatment of Bruce and Wall [BW] of the classification of singular cubic surfaces.

1.2. *Normal Forms for Quaternary Cubic Forms Defining Cubic Surfaces with Isolated Singularities.* Here, the normal form of f depends on the configuration of the singularities on the surface $f = 0$, and we obtain:

THEOREM 2. *The table overleaf lists the normal forms for quaternary cubic forms defining cubic surfaces with isolated singularities. The configuration of singularities on the respective surface is noted in the first column. Here, A_1 etc. refer to the classification of singularities (see e.g. [AGV], 242ff). Thus, $2A_1A_2$ means that there are two A_1-singularities and one A_2-singularity on the respective surface. It is assumed throughout that $l \in \mathbf{C}^*$.*

REMARK 2. The two different forms with a D_4-singularity are again distinguished by the geometry of the Hessian surface: The Hessian surface consists in the first case of a double plane and an irreducible quadric cone and in the second case of a double plane and two simple planes.

A_1	$lx_4(x_2^2 - x_1x_3) +$ $+ x_2\big(x_1 - (1+\rho_1)x_2 + \rho_1x_3\big)\big(x_1 - (\rho_2+\rho_3)x_2 + \rho_2\rho_3x_3\big)$, $\rho_i \in \mathbf{C} \setminus \{0,1\}$ *pairwise different*
$2A_1$	$lx_4(x_2^2 - x_1x_3) + x_2\big(x_1 - (1+\rho_1)x_2 + \rho_1x_3\big)(x_1 - \rho_2x_2)$, $\rho_i \in \mathbf{C} \setminus \{0,1\}$ *not equal*
$3A_1$	$lx_4(x_2^2 - x_1x_3) + x_2^2\big(x_1 - (1+\rho)x_2 + \rho x_3\big)$, $\rho \in \mathbf{C} \setminus \{0,1\}$
$4A_1$	$lx_4(x_2^2 - x_1x_3) + x_2^2(x_1 - 2x_2 + x_3)$
A_1A_2	$lx_4(x_2^2 - x_1x_3) + x_1x_2\big(x_1 - (1+\rho)x_2 + \rho x_3\big)$, $\rho \in \mathbf{C} \setminus \{0,1\}$
$2A_1A_2$	$lx_4(x_2^2 - x_1x_3) + x_2^2(x_1 - x_2)$
A_12A_2	$lx_4(x_2^2 - x_1x_3) + x_2^3$
A_1A_3	$lx_4(x_2^2 - x_1x_3) + x_1^2x_2 - x_1x_2^2$
$2A_1A_3$	$lx_4(x_2^2 - x_1x_3) + x_1x_2^2$
A_1A_4	$lx_4(x_2^2 - x_1x_3) + x_1^2x_2$
A_1A_5	$lx_4(x_2^2 - x_1x_3) + x_1^3$
A_2	$lx_4x_1x_2 - x_3(x_1^2 + x_2^2 + x_3^2 + \rho_1x_1x_3 + \rho_2x_2x_3)$, $\rho_1, \rho_2 \in \mathbf{C} \setminus \{-2, +2\}$
$2A_2$	$lx_4x_1x_2 - x_3(x_1^2 + x_3^2 + \rho x_1x_3)$, $\rho \in \mathbf{C} \setminus \{-2, +2\}$
$3A_2$	$lx_4x_1x_2 - x_3^3$
A_3	$lx_4x_1x_2 + x_1(x_1^2 - x_3^2) + \rho x_2(x_2^2 - x_3^2)$, $\rho \in \mathbf{C}^*$
A_4	$lx_4x_1x_2 + x_1^2x_3 + x_2(x_2^2 - x_3^2)$
A_5	$lx_4x_1x_2 + x_1^3 + x_2(x_2^2 - x_3^2)$
D_4^I	$lx_4x_1^2 + x_2^3 + x_3^3 + x_1x_2x_3$
D_4^{II}	$x_4x_1^2 + x_2^3 + x_3^3$
D_5	$x_4x_1^2 + x_1x_3^2 + x_2^2x_3$
E_6	$x_4x_1^2 + x_1x_3^2 + x_2^3$
$\widetilde{E}_6$	$x_1^3 + x_2^3 + x_3^3 - 3lx_1x_2x_3$, $l^3 \neq 1$

1.3. *Normal Forms for Quaternary Cubic Forms Defining Irreducible Cubic Surfaces with Non-Isolated Singularities.*

PROPOSITION 1. *The canonical forms for quaternary cubic forms defining irreducible cubic surfaces with non-isolated singularities are the following:*

Canonical form	*The surface* $f = 0$
$x_1^2x_3 + x_2^2x_4$	*Whitney's ruled surface*
$x_1^2x_3 + x_1x_2x_4 + x_2^3$	*Cayley's ruled surface*
$x_1x_3^2 + x_1x_2^2 + x_2^3$	*Cone over a nodal cubic*
$x_1^2x_3 + x_2^3$	*Cone over Neil's parabola*

REMARK 3. Cayley's ruled surface is actually a degeneration of Whitney's surface. Explicit constructions of those surfaces can be found in [Ha1], 330f, for Whitney's surface and in [Ha2], 80, for Cayley's surface.

1.4. *Normal Forms for Quaternary Cubic Forms Defining Reducible Cubic Surfaces.* Here, one obtains the following obvious result:

PROPOSITION 2. *A quaternary cubic form defining a reducible cubic surface can be brought into one of the following canonical forms:*

Canonical form	*The surface* $f = 0$
$(x_1 + x_2)(x_1x_2 + x_3x_4)$	*Non-sing. quadric w. transversal plane*
$x_1(x_1x_2 + x_3x_4)$	*Non-sing. quadric w. tangent plane*
$x_1(x_2^2 + x_3x_4)$	*Quadric cone w. transversal plane*
$x_2(x_2^2 + x_3x_4)$	*Cone over plane conic w. transversal line*
$x_3(x_2^2 + x_3x_4)$	*Cone over plane conic w. tangent*
$x_1x_2x_3$	*Three different planes*
$x_1x_2(x_1 + x_2)$	*Three different planes in a pencil*
$x_1^2x_2$	*Double plane and simple plane*
x_1^3	*Triple plane*

2. The Invariant Theory of Quaternary Cubic Forms

2.1. *Stable, Semistable and Nullforms.* The stable and semistable quaternary cubic forms and the quaternary cubic nullforms were determined by Hilbert [Hi] (for the definition of semistable and stable see [Ne], *nullform* means non-semistable form):

THEOREM 3. i) *A quaternary cubic form f is stable (resp. semistable) if and only if the surface $\{f=0\}$ has at most singularities of type A_1 (resp. A_2).*

ii) *A quaternary cubic form f is a nullform if and only if the surface $\{f=0\}$ has isolated singularities of type A_k $(k \geq 3)$, D_4, D_5, E_6, or $\widetilde{E}_6$, or if it has non-isolated singularities.*

2.2. *Degenerations of Orbits of Semistable Forms.* First, one observes that the semistable forms with closed orbit are precisely the forms whose associated cubic surfaces have three A_2-singularities. Applying Luna's slice theorem, one then computes the following table of degenerations where we characterize a form by the configuration of singularities on the corresponding cubic surface:

$$\begin{array}{ccccc} A_2 & & A_2A_1 & & 2A_1A_2 \\ & \searrow & & \searrow & \downarrow \\ & & 2A_2 & & 2A_2A_1 \\ & & & \searrow & \downarrow \\ & & & & 3A_2 \end{array}$$

The details can be found in [Sch1], 58ff.

2.3. *The Ring of Invariants.* Proofs of the following results can be found in the paper [Be]. We want to describe the ring $A := \mathbf{C}[S^3(\mathbf{C}^{4^\vee})]^{\mathrm{SL}_4(\mathbf{C})}$. This is the coordinate ring of the categorical quotient $S^3(\mathbf{C}^{4^\vee})/\!/\,\mathrm{SL}_4(\mathbf{C})$. It is the ring of polynomial expressions in the coefficients of cubic polynomials which are constant on all $\mathrm{SL}_4(\mathbf{C})$-orbits. In order to describe the ring A, we first introduce the following vector space

$$S := \left\{ r_1x_1^3 + r_2x_2^3 + r_3x_3^3 + r_4x_4^3 + r_5x_5^3 \mid \sum x_i = 0 \right\}.$$

On S, there is a natural action of the alternating group $\mathfrak{A}_5$, and $A \subset \mathbf{C}[S]^{\mathfrak{A}_5}$. This inclusion is constructed as follows: The group of automorphisms H of the Sylvester pentrahedron naturally acts on S, and it can be shown that the natural morphism $S/\!/H \longrightarrow S^3(\mathbf{C}^{4^\vee})/\!/\,\mathrm{SL}_4(\mathbf{C})$ is birational. This induces the inclusion $A \subset \mathbf{C}[S]^H$. Now, H is a finite group of order 480 obviously containing $\mathfrak{A}_5$. Denote by σ_i, $i = 1,2,3,4,5$, and v the i-th symmetric function and the Vandermonde determinant in the r_i. Then $\mathbf{C}[S]^{\mathfrak{A}_5} = \mathbf{C}[\sigma_1,\dots,\sigma_5,v]$.

THEOREM 4. *The ring of invariants A is the subring of $\mathbf{C}[S]^{\mathfrak{A}_5}$ generated by the following invariant polynomials*

$$I_8 := \sigma_4^2 - 4\sigma_3\sigma_5, \quad I_{16} := \sigma_5^3\sigma_1, \quad I_{24} := \sigma_5^4\sigma_4,$$
$$I_{32} := \sigma_5^6\sigma_2, \quad I_{40} := \sigma_5^8, \quad I_{100} := \sigma_5^{18}v,$$

which satisfy a relation

$$I_{100}^2 = P(I_8, I_{16}, I_{24}, I_{32}, I_{40}).$$

2.4. *The Discriminant.* Using techniques from the paper [BC], one obtains the following

PROPOSITION 3. *The discriminant of quaternary cubic forms is given by the formula*

$$\Delta = (I_8^2 - 64I_{16})^2 - 2^{11}(I_8I_{24} + 8I_{32}).$$

2.5. *Moduli Spaces of Cubic Surfaces.* Define $\overline{\mathcal{M}}$ to be the hypersurface $\{I_{100}^2 - P(I_8, I_{16}, I_{24}, I_{32}, I_{40}) = 0\}$ in the weighted projective space $\mathbf{P}(8,16,24,32,40) = \mathbf{P}(1,2,3,4,5)$. Then $\mathcal{M} := \overline{\mathcal{M}} \setminus \{\Delta = 0\}$ is a moduli space for non-singular cubic surfaces. On the other hand, every non-singular cubic surface can be obtained as the blow up of $\mathbf{P}_2$ in six points in general position. The sextuples of points in general position form an open subset $\mathcal{U} \subset S^6\mathbf{P}_2$ of the sixth symmetric power of $\mathbf{P}_2$. Furthermore, there is an action of $\mathrm{PGL}_3(\mathbf{C})$ on $\mathcal{U}$, and the geometric quotient $\mathcal{N} := \mathcal{U}/\!/\,\mathrm{PGL}_3(\mathbf{C})$ does exist [Is]. By [Is], §6, $\mathcal{N}$ is a coarse moduli space for pairs (X,L) consisting of a cubic surface X and a globally generated line bundle L which defines a blow down $X \longrightarrow \mathbf{P}_2$. Forgetting the line bundle L provides us with a morphism $\mathcal{N} \longrightarrow \mathcal{M}$, so that there is a surjection $f\colon \mathcal{U} \longrightarrow \mathcal{M}$. Hence, we can view the invariants of quaternary cubic forms as regular functions on $\mathcal{U}$. This relates the geometry of the cubic surface to the set of six points. One obtains, e.g., PROPOSITION 4. *The set of sextuples in $\mathcal{U}$ whose associated cubic surface is given by an equation which is not a (nondegenerate) Sylvestrian pentahedral form is the Zariski-closed subset* $\{f^* I_{40} = 0\}$.

Of course, a better understanding of the geometric meaning of the other invariants should allow to extend this result.

II. CUBIC FORMS OF PROJECTIVE THREEFOLDS

1. PRELIMINARIES

For the convenience of the reader, we have collected the crucial theorems which we will use in the construction of our examples.

1.1. *The Lefschetz Theorem on Hyperplane Sections.* We summarize Bertini's Theorem and Lefschetz' Theorem in:

THEOREM 5. *Let Y be a projective manifold, L a very ample line bundle on Y, and $X := Z(s)$ the zero-set of a general section $s \in H^0(X, L)$. Then X is a manifold (connected if* $\dim Y \geq 2$*), and the inclusion $\iota\colon X \hookrightarrow Y$ induces isomorphisms*

$$\iota^* : H^i(Y, \mathbf{Z}) \longrightarrow H^i(X, \mathbf{Z}), \qquad i = 1, \ldots, \dim Y - 2;$$
$$\iota_* : \pi_i(X) \longrightarrow \pi_i(Y), \qquad i = 1, \ldots, \dim Y - 2.$$

Proof. [La], Th. 3.6.7 & Th. 8.1.1. □

1.2. *Formulas for Blow Ups.* A very simple way to obtain a new manifold from a given one is the blow up in a point or along a smooth curve. The cup form behaves as follows (we will suppose for simplicity that $H^2(Y, \mathbf{Z})$ is without torsion):

THEOREM 6. i) *Let $\sigma\colon X \longrightarrow Y$ be the blow up of Y in a point. Let $q(x_1, \ldots, x_n)$ be the cubic polynomial which describes the cup form of Y w. r. t. the basis $(\kappa_1, \ldots, \kappa_n)$ of $H^2(Y, \mathbf{Z})$. If $h_0 \in H^2(X, \mathbf{Z})$ is the cohomology class of the exceptional divisor, then $(h_0, \sigma^*\kappa_1, \ldots, \sigma^*\kappa_n)$ is a basis of $H^2(Y, \mathbf{Z})$ w. r. t. which the cup form of X is given by*

$$x_0^3 + q(x_1, \ldots, x_n).$$

ii) *Let $C \subset Y$ be a smooth curve, and $\sigma\colon X \longrightarrow Y$ be the blow up of Y along this curve. Using the same notation as in* i), *the cup form of X is described by the polynomial*

$$q(x_1, \dots, x_n) - 3 \cdot \Big(\sum_{i=1}^{n} (C \,.\, \kappa_i) x_i x_0^2\Big) - \deg_C(N_{C/Y}) x_0^3 \,.$$

Here, $C \,.\, \kappa_i$ stands for the evaluation of the homology class of C on κ_i, and $N_{C/Y}$ is the normal bundle of C in Y.

Proof. This follows easily from [GH], p.602ff. □

1.3. *Complete Intersections in Products of Projective Spaces.*

Let $\mathbf{P}_{n_1} \times \dots \times \mathbf{P}_{n_r}$ be a product of projective spaces. Write $\mathcal{O}(a_1, \dots, a_r)$ for the invertible sheaf $\pi_1^* \mathcal{O}_{\mathbf{P}_{n_1}}(a_1) \otimes \dots \otimes \pi_r^* \mathcal{O}_{\mathbf{P}_{n_r}}(a_r)$. Here, π_i is the projection onto the i-th factor. If all the a_i's are positive, this sheaf is very ample. A section in it is given by a multihomogeneous polynomial of multidegree $(a_1, \dots, a_r)$. We denote by

$$\begin{bmatrix} \mathbf{P}_{n_1} & | & a_1^1 & \dots & a_1^m \\ \vdots & & \vdots & & \vdots \\ \mathbf{P}_{n_r} & | & a_r^1 & \dots & a_r^m \end{bmatrix}$$

the family of zero sets of sections of the sheaf

$$\mathcal{O}(a_1^1, \dots, a_r^1) \oplus \dots \oplus \mathcal{O}(a_1^m, \dots, a_r^m) \,.$$

The members of this family are complete intersections of m hypersurfaces. An iterated application of Theorem 5 shows that a general member X of such a family is smooth and simply connected and that $(h_1, \dots, h_m)$ with $h_i := \pi_i^*(c_1(\mathcal{O}_{\mathbf{P}_{n_i}}(1)))$ is a basis for $H^2(X, \mathbf{Z})$.

2. A Projective Threefold with a Nodal Cubic as Cup Form

Let Y be a smooth member of the family $\begin{bmatrix} \mathbf{P}_4 & | & 1 & 2 \\ \mathbf{P}_1 & | & 1 & 1 \end{bmatrix}$. We first compute the cup form of Y. Let $(\widetilde{h}_1, \widetilde{h}_2)$ be the canonical basis of $H^2(\mathbf{P}_4 \times \mathbf{P}_1, \mathbf{Z})$, and (h_1, h_2) be the basis of $H^2(X, \mathbf{Z})$ as described in 1.3. We compute, e.g.,

$$h_1^2 h_2 = \widetilde{h}_1^2 \widetilde{h}_2 (\widetilde{h}_1 + \widetilde{h}_2)(2\widetilde{h}_1 + \widetilde{h}_2) = 2\widetilde{h}_1^4 \widetilde{h}_2 = 2 \,.$$

Here we have written the cup product followed by evaluation on the fundamental class as multiplication. The cup form of Y is given by the polynomial

$$3x_1^3 + 6x_1^2 x_2 \,.$$

Y contains four smooth curves $C_i \cong \mathbf{P}_1$, $i = 1, \ldots, 4$, such that $C_i . h_1 = 0$, $C_i . h_2 = 1$, and $N_{C_i/X} \cong \mathcal{O}_{\mathbf{P}_1}(-1) \oplus \mathcal{O}_{\mathbf{P}_1}(-1)$. To see this, observe that Y is defined by two equations $l_0 \cdot x_0 + l_1 \cdot x_1 = 0$ and $q_0 \cdot x_0 + q_1 \cdot x_1 = 0$. Here, x_0 and x_1 are the homogeneous coordinates of $\mathbf{P}_1$ and l_0, l_1 and q_0, q_1 are linear and quadratic homogeneous polynomials in 5 variables (the homogeneous coordinates of $\mathbf{P}_4$). It is easily computed that the image of Y under the projection to $\mathbf{P}_4$ is the hypersurface $\widetilde{Y} := \{l_0 q_1 - l_1 q_0 = 0\}$. For a generic choice of l_0, l_1, q_0, q_1, the set $S := Z(l_0, l_1, q_0, q_1)$ consists of 4 points (Thm. 5). It is obvious that the projection $Y \longrightarrow \widetilde{Y}$ is an isomorphism above $\widetilde{Y} \setminus S$ and that the fibre above a point in S is of the type $\{s\} \times \mathbf{P}_1$. The description of the normal bundle is a consequence of this. Let X be the blow up of Y in one of these curves. By Theorem 6, the cup form of X is given by the polynomial

$$3x_1^3 + 6x_1^2 x_2 - 3x_0^2 x_2 + 2x_0^3 .$$

This defines an irreducible plane cubic with a node.

3. Quaternary Cubic Forms that are Cup Forms of Projective Algebraic Manifolds

On the one hand, we know by [OV], Prop. 16, that cubic forms whose Hessian vanishes identically cannot occur as cup forms of projective threefolds. The Hessian of a quaternary cubic form f vanishes identically if and only if the surface $f = 0$ is a cone over a plane cubic curve. On the other hand, we have collected a number of families in which we find cup forms of simply connected projective threefolds. There are some families which are not covered by these two results, for them the problem of realizability remains unsolved.

THEOREM 7. *There are polynomials occurring as cup forms of projective algebraic manifolds in the following families of non-singular forms:*

$$(*), \quad (*_1), \quad (*_2), \quad (*_4) \text{ and } (*_5),$$

and in the following families of forms defining surfaces with isolated singularities:

$$(A_1),\ (2A_1),\ (3A_1),\ (4A_1),\ (2A_1A_2),\ (A_2),\ (2A_2),\ (3A_2) \text{ and } (D_4^{II}).$$

Furthermore, forms which define in $\mathbf{P}_3$ the union of a non-singular quadric with a transversal plane, or the union of a quadric cone with a transversal plane can be realized as cup forms of projective algebraic manifolds.

We prove this result by giving examples. The cup form of a complete intersection in a product of projective spaces is easily computed as in Section 2. Furthermore, we will mainly use blow up constructions, hence one has to use the formulas of Theorem 6. In some cases, we will give further details.

3.1. *Sylvestrian Pentahedral Forms.* Consider in $\mathbf{P}_2 \times \mathbf{P}_2 \times \mathbf{P}_2 \times \mathbf{P}_2$ a smooth member X of the family $\begin{bmatrix} \mathbf{P}_2 & | & 1\;1\;1\;1\;1 \\ \mathbf{P}_2 & | & 1\;1\;1\;1\;1 \\ \mathbf{P}_2 & | & 1\;1\;1\;1\;1 \\ \mathbf{P}_2 & | & 1\;1\;1\;1\;1 \end{bmatrix}$. The cup form of X is described by the following polynomial:

$$\begin{aligned} 90x_1^2x_2 &+ 90x_1^2x_3 + 90x_1^2x_4 + 90x_1x_2^2 + 90x_1x_3^2 + 90x_1x_4^2 \\ &+ 90x_2^2x_3 + 90x_2^2x_4 + 90x_2x_3^2 + 90x_2x_4^2 + 90x_3^2x_4 + 90x_3x_4^2 \\ &+ 360x_1x_2x_3 + 360x_1x_2x_4 + 360x_1x_3x_4 + 360x_2x_3x_4 \\ &= -30(x_2+x_3+x_4)^3 - 30(x_1+x_3+x_4)^2 - 30(x_1+x_2+x_4)^3 \\ &\qquad\qquad - 30(x_1+x_2+x_3)^3 + 90(x_1+x_2+x_3+x_4)^3 . \end{aligned}$$

3.2. *Diagonal Forms.* Let $X = \widehat{\mathbf{P}}_3(p_1, p_2, p_3)$ be the blow up of $\mathbf{P}_3$ in three points. By Theorem 6, the cup form of X is then given by

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 .$$

3.3. *Non-Equianharmonic Forms.* We begin by constructing a manifold Y with $b_2 = 3$ whose cup form is a non-equianharmonic ternary cubic form. X will then be defined as the blow up of Y in one point. Suppose $Z \subset \mathbf{P}_2 \times \mathbf{P}_2$ is a smooth member of the linear system $\begin{bmatrix} \mathbf{P}_2 & | & 1 \\ \mathbf{P}_2 & | & 1 \end{bmatrix}$, and $C = Z(s)$ is the zero locus of a general section $s \in H^0(Z, \mathcal{O}_Z(1,0) \oplus \mathcal{O}_Z(1,1))$. Let Y be the blow up of Z along the curve C. In order to apply Theorem 6, we will have to compute the intersections $C . h_i$, $i = 1, 2$, where (h_1, h_2) is the canonical basis of $H^2(Z, \mathbf{Z})$. To do this, we observe that the cohomology class associated to C is just $h_1 \cup (h_1 + h_2)$. Thus, for example,

$$h_1 . C = (h_1 \cup h_1 \cup (h_1 + h_2))[X] = h_1^3 + h_1^2 h_2 = h_1^2 h_2 = 1 .$$

We also have to compute $\deg_C(N_{C/Z})$. This number is given by

$$\big((2h_1 + h_2) \cup h_1 \cup (h_1 + h_2)\big)[Z] = 3h_1^2 h_2 + h_1 h_2^2 = 4 .$$

The cup form of Y is given by

$$f := 3x_1^2 x_2 + 3x_1 x_2^2 - 3x_1 x_3^2 - 6x_2 x_3^2 - 4x_3^3 .$$

This describes a smooth plane cubic, and we must show that its j-invariant does not vanish. For this, it suffices to verify that the Aronhold S-invariant is non-zero [St, p.173]. The invariant S takes the value 3 on f [St, Prop. 4.4.7].

3.4. *Non-Singular Forms whose Hessians have Seven Singular Points.* We look at a smooth member Y of the family $\left[\begin{smallmatrix}\mathbf{P}_2 & | & 1\\ \mathbf{P}_2 & | & 1\end{smallmatrix}\right]$. Let $C_1 = Z(s_1)$ and $C_2 = Z(s_2)$ be two disjoint smooth curves where $s_1, s_2 \in H^0(Y, \mathcal{O}_Y(1,0) \oplus \mathcal{O}_Y(1,1))$ are chosen generically. The blow up X of Y along those curves has the following cup form:

$$3x_1^2x_2 + 3x_1x_2^2 - 3x_1x_3^2 - 6x_2x_3^2 - 4x_3^3 - 3x_1x_4^2 - 6x_2x_4^2 - 4x_4^3 .$$

This polynomial can be written as:

$$\begin{aligned}&-\tfrac{3}{4}x_1^3 - \tfrac{1}{16}(x_1 + 2x_2 + 4x_3)^3 - \tfrac{1}{16}(x_1 + 2x_2 + 4x_4)^3\\ &+ 3(x_1 + 2x_2)^2\left(\tfrac{1}{4}x_1 - \tfrac{1}{12}(x_1 + 2x_2) + \tfrac{1}{16}(x_1 + 2x_2 + 4x_3) + \tfrac{1}{16}(x_1 + 2x_2 + 4x_4)\right) .\end{aligned}$$

3.5. *Non-Singular Forms whose Hessians have Four Singular Points.* Let Y be again a smooth member of $\left[\begin{smallmatrix}\mathbf{P}_2 & | & 1\\ \mathbf{P}_2 & | & 1\end{smallmatrix}\right]$ and choose two disjoint smooth curves $C_1 = Z(s_1)$ and $C_2 = Z(s_2)$, where $s_1, s_2 \in H^0(Y, \mathcal{O}_Y(1,0) \oplus \mathcal{O}_Y(0,1))$ are general sections. The blow up X of Y along those two curves has the polynomial

$$3x_1^2x_2 + 3x_1x_2^2 - 3x_1x_3^2 - 3x_2x_3^2 - 2x_3^3 - 3x_1x_4^2 - 3x_2x_4^2 - 2x_4^3$$

as the cup form, and one checks that its Hessian defines a surface with singularities in the points $[1 : -1 : 0 : 0]$, $[0 : 0 : 0 : 1]$, $[0 : 0 : 1 : 0]$, and $[0 : 0 : -1 : 1]$.

3.6. (A_1). Suppose Y is a smooth member of $\left[\begin{smallmatrix}\mathbf{P}_4 & | & 2 & 1\\ \mathbf{P}_1 & | & 1 & 1\end{smallmatrix}\right]$. As we have seen before, Y contains four curves of the type $\{p\} \times \mathbf{P}_1$ with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Define X as the blow up of Y along two of those curves. The cup form of X is then described by the polynomial

$$3x_1^3 + 6x_1^2x_2 - 3x_2x_3^2 - 3x_2x_4^2 + 2x_3^3 + 2x_4^3$$

which defines a surface in $\mathbf{P}_3$ with an A_1-singularity in $[0 : 1 : 0 : 0]$.

3.7. $(2A_1)$. Let X be a smooth member of $\left[\begin{array}{c|ccc} \mathbf{P}_1 & 1 & 1 & 1 \\ \mathbf{P}_1 & 1 & 1 & 1 \\ \mathbf{P}_2 & 1 & 1 & 1 \\ \mathbf{P}_2 & 1 & 1 & 1 \end{array}\right]$. Its cup form is

$$9x_1x_3^2 + 9x_1x_4^2 + 9x_2x_3^2 + 9x_2x_4^2 + 18x_3^2x_4 + 18x_3x_4^2 \\ + 18x_1x_2x_3 + 18x_1x_2x_4 + 36x_1x_3x_4 + 36x_2x_3x_4 .$$

The surface in $\mathbf{P}_3$ defined by this polynomial has two A_1-singularities in $[1:0:0:0]$ and $[0:1:0:0]$.

3.8. $(3A_1)$. Blow up $\mathbf{P}_1 \times \mathbf{P}_1 \times \mathbf{P}_1$ in $C = Z(s)$, where $s \in H^0(\mathcal{O}(1,1,0) \oplus \mathcal{O}(0,1,1))$ is a generically chosen section in order to obtain a manifold X with cup form

$$6x_1x_2x_3 - 3x_1x_4^2 - 3x_2x_4^2 - 3x_3x_4^2 - 4x_4^3 .$$

The corresponding cubic surface has three A_1-singularities in the points $[1:0:0:0]$, $[0:1:0:0]$, and $[0:0:1:0]$.

3.9. $(4A_1)$. Let $V \subset \mathbf{P}_1 \times \mathbf{P}_1 \times \mathbf{P}_1 \times \mathbf{P}_1$ be a smooth divisor in $|\mathcal{O}(a_1, a_2, a_3, a_4)|$, $a_i > 0$, $i = 1, 2, 3, 4$. The cup form of this manifold is described by the polynomial

$$6a_1x_2x_3x_4 + 6a_2x_1x_3x_4 + 6a_3x_1x_2x_4 + 6a_4x_1x_2x_3 ,$$

which defines a surface in $\mathbf{P}_3$ with A_1-singularities in $[1:0:0:0]$, $[0:1:0:0]$, $[0:0:1:0]$, and $[0:0:0:1]$.

3.10. (A_2). Let Y be a smooth complete intersection in the family $\left[\begin{array}{c|cc} \mathbf{P}_4 & 2 & 1 \\ \mathbf{P}_1 & 1 & 1 \end{array}\right]$. This time, blow up Y along a curve of the type $\{p\} \times \mathbf{P}_1$ and in a point. The resulting manifold X has the cup form

$$3x_1^3 + 6x_1^2x_2 - 3x_2x_3^2 + 2x_3^3 + x_4^3 .$$

This polynomial defines a cubic surface with an A_2-singularity in $[0:1:0:0]$.

3.11. $(2A_2)$. Let Y be a projective algebraic threefold with $b_2 = 3$ and cup form $q(x_1, x_2, x_3)$ and suppose that q defines a smooth conic with a transversal line (this happens, e.g., when Y is a $\mathbf{P}_1$-bundle over some surface). The cup form of Y blown up in one point is then

$$q(x_1, x_2, x_3) + x_4^3 .$$

3.12. ($3A_2$). Let $X = (\mathbf{P}_1 \times \mathbf{P}_1 \times \mathbf{P}_1)^\wedge(p)$ be the blow up of $\mathbf{P}_1 \times \mathbf{P}_1 \times \mathbf{P}_1$ in the point p. The cup form of X is

$$x_4^3 + 6x_1x_2x_3 .$$

3.13. ($2A_1A_2$). Consider the curve $C = Z(s) \subset \mathbf{P}_1 \times \mathbf{P}_1 \times \mathbf{P}_1$ where $s \in H^0(\mathcal{O}(1,1,0) \oplus \mathcal{O}(0,0,1))$ is a general section, and let X be the blow up of $\mathbf{P}_1 \times \mathbf{P}_1 \times \mathbf{P}_1$ along C. The cup form

$$6x_1x_2x_3 - 3x_1x_4^2 - 3x_2x_4^2 - 2x_4^3$$

of X defines a cubic surface with A_1-singularities in $[1:0:0:0]$ and $[0:1:0:0]$ and an A_2-singularity in $[0:0:1:0]$.

3.14. (D_4^{II}). Let $X := \widehat{\mathbf{P}_1 \times \mathbf{P}_2}(p_1, p_2)$ be the blow up of $\mathbf{P}_1 \times \mathbf{P}_2$ in the points p_1 and p_2. Its cup form is described by the polynomial

$$3x_1x_2^2 + x_3^3 + x_4^3 .$$

This polynomial is the equation of a cubic surface with a D_4-singularity in $[1:0:0:0]$.

3.15. *A Non-Singular Quadric with a Transversal Plane.* Manifolds with such cup forms may be obtained as suitable $\mathbf{P}_1$-bundles over surfaces. Indeed, let Y be a smooth surface with $b_2 = 3$. W. r. t. a suitable basis (h_1, h_2, h_3) of $H^2(Y, \mathbf{Z})$, its cup form is given by $x_1^2 + x_2^2 + x_3^2$. Now, let E be a vector bundle of rank 2 such that $c_1^2(E) - c_2(E) \neq 0$. Let $X := \mathbf{P}(E) \xrightarrow{\pi} Y$ and choose $(\pi^*h_1, \pi^*h_2, \pi^*h_3, c_1(\mathcal{O}_X(1)))$ as a basis of $H^2(X, \mathbf{Z})$. Then, by [OV], Prop. 15, the cup form of X is given by

$$(c_1^2(E) - c_2(E))x_4^3 + x_4(x_1^2 + x_2^2 + x_3^2) .$$

3.16. *A Quadric Cone with a Transversal Plane.* Let Y be a simply connected surface with $b_2 = 3$ and torsion free homology. The cup form of Y is given by a quadratic polynomial $q(x_1, x_2, x_3)$ defining a smooth conic. Thus, the cup form of $Y \times \mathbf{P}_1$ is given by

$$x_4\, q(x_1, x_2, x_3) .$$

4. Real Cubic Forms which are not Cup Forms of Projective Algebraic Manifolds

In the paper [Sch2], the author investigated the restrictions on the real cubic forms of projective manifolds imposed by the so called Hodge-Riemann bilinear relations:

THEOREM 8. *Let X be a Kählerian threefold and $h \in H^2(X,\mathbf{R})$ be a Kähler class. Then the map*

$$\begin{array}{rccc} \langle\,.\,,.\,\rangle : & H^2(X,\mathbf{R}) \times H^2(X,\mathbf{R}) & \longrightarrow & \mathbf{R} \\ & (a,b) & \longmapsto & a \cup b \cup h \end{array}$$

is a non-degenerate, symmetric bilinear form of signature $(2h^{2,0}+1, h^{1,1}-1)$.

One can restate this theorem in such a form as to obtain – at least in theory – some explicit inequalities in the coefficients of cubic polynomials which are satisfied by the cup forms of Kählerian and hence projective algebraic threefolds. The main result of [Sch2] is

THEOREM 9. *For $n \geq 4$, the polynomial*

$$x_0 \left(\frac{4-n}{4} x_0^2 - 3x_1^2 - \cdots - 3x_n^2 \right)$$

cannot occur as the (real) cup form of a projective algebraic threefold with $b_1 = 0$ and $b_3 = 0$.

As a corollary, one obtains the following generalization of a result of Campana and Peternell [CP]:

THEOREM 10. *For $n \geq 4$, twistor spaces over $\sharp_{i=1}^n \mathbf{P}_2$ are not homeomorphic to projective algebraic threefolds.*

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