II. Cubic Forms of Projective Threefolds

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PROPOSITION 4. *The set of sextuples in $\mathcal{U}$ whose associated cubic surface is given by an equation which is not a (nondegenerate) Sylvestrian pentahedral form is the Zariski-closed subset* $\{f^*I_{40} = 0\}$.

Of course, a better understanding of the geometric meaning of the other invariants should allow to extend this result.

II. CUBIC FORMS OF PROJECTIVE THREEFOLDS

1. PRELIMINARIES

For the convenience of the reader, we have collected the crucial theorems which we will use in the construction of our examples.

1.1. *The Lefschetz Theorem on Hyperplane Sections.* We summarize Bertini's Theorem and Lefschetz' Theorem in:

THEOREM 5. *Let Y be a projective manifold, L a very ample line bundle on Y, and $X := Z(s)$ the zero-set of a general section $s \in H^0(X, L)$. Then X is a manifold (connected if* $\dim Y \geq 2$*), and the inclusion $\iota\colon X \hookrightarrow Y$ induces isomorphisms*

$$\begin{aligned} \iota^* &: H^i(Y, \mathbf{Z}) \longrightarrow H^i(X, \mathbf{Z}), && i = 1, \dots, \dim Y - 2; \\ \iota_* &: \pi_i(X) \longrightarrow \pi_i(Y), && i = 1, \dots, \dim Y - 2. \end{aligned}$$

Proof. [La], Th. 3.6.7 & Th. 8.1.1. □

1.2. *Formulas for Blow Ups.* A very simple way to obtain a new manifold from a given one is the blow up in a point or along a smooth curve. The cup form behaves as follows (we will suppose for simplicity that $H^2(Y, \mathbf{Z})$ is without torsion):

THEOREM 6. i) *Let $\sigma\colon X \longrightarrow Y$ be the blow up of Y in a point. Let $q(x_1, \dots, x_n)$ be the cubic polynomial which describes the cup form of Y w. r. t. the basis $(\kappa_1, \dots, \kappa_n)$ of $H^2(Y, \mathbf{Z})$. If $h_0 \in H^2(X, \mathbf{Z})$ is the cohomology class of the exceptional divisor, then $(h_0, \sigma^*\kappa_1, \dots, \sigma^*\kappa_n)$ is a basis of $H^2(Y, \mathbf{Z})$ w. r. t. which the cup form of X is given by*

$$x_0^3 + q(x_1, \dots, x_n).$$

ii) *Let $C \subset Y$ be a smooth curve, and $\sigma\colon X \longrightarrow Y$ be the blow up of Y along this curve. Using the same notation as in* i), *the cup form of X is described by the polynomial*

$$q(x_1, \dots, x_n) - 3 \cdot \Big(\sum_{i=1}^{n} (C \,.\, \kappa_i) x_i x_0^2\Big) - \deg_C(N_{C/Y}) x_0^3 \,.$$

Here, $C \,.\, \kappa_i$ stands for the evaluation of the homology class of C on κ_i, and $N_{C/Y}$ is the normal bundle of C in Y.

Proof. This follows easily from [GH], p.602ff. $\square$

1.3. *Complete Intersections in Products of Projective Spaces.*

Let $\mathbf{P}_{n_1} \times \cdots \times \mathbf{P}_{n_r}$ be a product of projective spaces. Write $\mathcal{O}(a_1, \dots, a_r)$ for the invertible sheaf $\pi_1^* \mathcal{O}_{\mathbf{P}_{n_1}}(a_1) \otimes \cdots \otimes \pi_r^* \mathcal{O}_{\mathbf{P}_{n_r}}(a_r)$. Here, π_i is the projection onto the i-th factor. If all the a_i's are positive, this sheaf is very ample. A section in it is given by a multihomogeneous polynomial of multidegree $(a_1, \dots, a_r)$. We denote by

$$\begin{bmatrix} \mathbf{P}_{n_1} & | & a_1^1 & \dots & a_1^m \\ \vdots & & \vdots & & \vdots \\ \mathbf{P}_{n_r} & | & a_r^1 & \dots & a_r^m \end{bmatrix}$$

the family of zero sets of sections of the sheaf

$$\mathcal{O}(a_1^1, \dots, a_r^1) \oplus \cdots \oplus \mathcal{O}(a_1^m, \dots, a_r^m) \,.$$

The members of this family are complete intersections of m hypersurfaces. An iterated application of Theorem 5 shows that a general member X of such a family is smooth and simply connected and that $(h_1, \dots, h_m)$ with $h_i := \pi_i^*(c_1(\mathcal{O}_{\mathbf{P}_{n_i}}(1)))$ is a basis for $H^2(X, \mathbf{Z})$.

2. A Projective Threefold with a Nodal Cubic as Cup Form

Let Y be a smooth member of the family $\begin{bmatrix} \mathbf{P}_4 & | & 1 & 2 \\ \mathbf{P}_1 & | & 1 & 1 \end{bmatrix}$. We first compute the cup form of Y. Let $(\widetilde{h}_1, \widetilde{h}_2)$ be the canonical basis of $H^2(\mathbf{P}_4 \times \mathbf{P}_1, \mathbf{Z})$, and (h_1, h_2) be the basis of $H^2(X, \mathbf{Z})$ as described in 1.3. We compute, e.g.,

$$h_1^2 h_2 = \widetilde{h}_1^2 \widetilde{h}_2 (\widetilde{h}_1 + \widetilde{h}_2)(2\widetilde{h}_1 + \widetilde{h}_2) = 2\widetilde{h}_1^4 \widetilde{h}_2 = 2 \,.$$

Here we have written the cup product followed by evaluation on the fundamental class as multiplication. The cup form of Y is given by the polynomial

$$3x_1^3 + 6x_1^2 x_2 \,.$$

Y contains four smooth curves $C_i \cong \mathbf{P}_1$, $i = 1, \ldots, 4$, such that $C_i \,.\, h_1 = 0$, $C_i \,.\, h_2 = 1$, and $N_{C_i/X} \cong \mathcal{O}_{\mathbf{P}_1}(-1) \oplus \mathcal{O}_{\mathbf{P}_1}(-1)$. To see this, observe that Y is defined by two equations $l_0 \cdot x_0 + l_1 \cdot x_1 = 0$ and $q_0 \cdot x_0 + q_1 \cdot x_1 = 0$. Here, x_0 and x_1 are the homogeneous coordinates of $\mathbf{P}_1$ and l_0, l_1 and q_0, q_1 are linear and quadratic homogeneous polynomials in 5 variables (the homogeneous coordinates of $\mathbf{P}_4$). It is easily computed that the image of Y under the projection to $\mathbf{P}_4$ is the hypersurface $\widetilde{Y} := \{l_0 q_1 - l_1 q_0 = 0\}$. For a generic choice of l_0, l_1, q_0, q_1, the set $S := Z(l_0, l_1, q_0, q_1)$ consists of 4 points (Thm. 5). It is obvious that the projection $Y \longrightarrow \widetilde{Y}$ is an isomorphism above $\widetilde{Y} \setminus S$ and that the fibre above a point in S is of the type $\{s\} \times \mathbf{P}_1$. The description of the normal bundle is a consequence of this. Let X be the blow up of Y in one of these curves. By Theorem 6, the cup form of X is given by the polynomial

$$3x_1^3 + 6x_1^2 x_2 - 3x_0^2 x_2 + 2x_0^3 \,.$$

This defines an irreducible plane cubic with a node.

3. Quaternary Cubic Forms that are Cup Forms of Projective Algebraic Manifolds

On the one hand, we know by [OV], Prop. 16, that cubic forms whose Hessian vanishes identically cannot occur as cup forms of projective threefolds. The Hessian of a quaternary cubic form f vanishes identically if and only if the surface $f = 0$ is a cone over a plane cubic curve. On the other hand, we have collected a number of families in which we find cup forms of simply connected projective threefolds. There are some families which are not covered by these two results, for them the problem of realizability remains unsolved.

THEOREM 7. *There are polynomials occurring as cup forms of projective algebraic manifolds in the following families of non-singular forms:*

$$(*), \quad (*_1), \quad (*_2), \quad (*_4) \ \textit{and} \ (*_5)\,,$$

and in the following families of forms defining surfaces with isolated singularities:

$$(A_1),\ (2A_1),\ (3A_1),\ (4A_1),\ (2A_1A_2),\ (A_2),\ (2A_2),\ (3A_2) \ \textit{and} \ (D_4^{II})\,.$$

Furthermore, forms which define in $\mathbf{P}_3$ *the union of a non-singular quadric with a transversal plane, or the union of a quadric cone with a transversal plane can be realized as cup forms of projective algebraic manifolds.*

We prove this result by giving examples. The cup form of a complete intersection in a product of projective spaces is easily computed as in Section 2. Furthermore, we will mainly use blow up constructions, hence one has to use the formulas of Theorem 6. In some cases, we will give further details.

3.1. *Sylvestrian Pentahedral Forms.* Consider in $\mathbf{P}_2 \times \mathbf{P}_2 \times \mathbf{P}_2 \times \mathbf{P}_2$ a smooth member X of the family $\left[\begin{array}{c|ccccc} \mathbf{P}_2 & 1 & 1 & 1 & 1 & 1 \\ \mathbf{P}_2 & 1 & 1 & 1 & 1 & 1 \\ \mathbf{P}_2 & 1 & 1 & 1 & 1 & 1 \\ \mathbf{P}_2 & 1 & 1 & 1 & 1 & 1 \end{array}\right]$. The cup form of X is described by the following polynomial:

$$\begin{aligned} &90x_1^2x_2 + 90x_1^2x_3 + 90x_1^2x_4 + 90x_1x_2^2 + 90x_1x_3^2 + 90x_1x_4^2 \\ &\quad + 90x_2^2x_3 + 90x_2^2x_4 + 90x_2x_3^2 + 90x_2x_4^2 + 90x_3^2x_4 + 90x_3x_4^2 \\ &\quad + 360x_1x_2x_3 + 360x_1x_2x_4 + 360x_1x_3x_4 + 360x_2x_3x_4 \\ &\quad = -30(x_2+x_3+x_4)^3 - 30(x_1+x_3+x_4)^2 - 30(x_1+x_2+x_4)^3 \\ &\qquad\qquad - 30(x_1+x_2+x_3)^3 + 90(x_1+x_2+x_3+x_4)^3 . \end{aligned}$$

3.2. *Diagonal Forms.* Let $X = \widehat{\mathbf{P}}_3(p_1, p_2, p_3)$ be the blow up of $\mathbf{P}_3$ in three points. By Theorem 6, the cup form of X is then given by

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 .$$

3.3. *Non-Equianharmonic Forms.* We begin by constructing a manifold Y with $b_2 = 3$ whose cup form is a non-equianharmonic ternary cubic form. X will then be defined as the blow up of Y in one point. Suppose $Z \subset \mathbf{P}_2 \times \mathbf{P}_2$ is a smooth member of the linear system $\left[\begin{array}{c|c} \mathbf{P}_2 & 1 \\ \mathbf{P}_2 & 1 \end{array}\right]$, and $C = Z(s)$ is the zero locus of a general section $s \in H^0(Z, \mathcal{O}_Z(1,0) \oplus \mathcal{O}_Z(1,1))$. Let Y be the blow up of Z along the curve C. In order to apply Theorem 6, we will have to compute the intersections $C \,.\, h_i$, $i = 1, 2$, where (h_1, h_2) is the canonical basis of $H^2(Z, \mathbf{Z})$. To do this, we observe that the cohomology class associated to C is just $h_1 \cup (h_1 + h_2)$. Thus, for example,

$$h_1 \,.\, C = (h_1 \cup h_1 \cup (h_1 + h_2))[X] = h_1^3 + h_1^2h_2 = h_1^2h_2 = 1 .$$

We also have to compute $\deg_C(N_{C/Z})$. This number is given by

$$\big((2h_1 + h_2) \cup h_1 \cup (h_1 + h_2)\big)[Z] = 3h_1^2h_2 + h_1h_2^2 = 4 .$$

The cup form of Y is given by

$$f := 3x_1^2x_2 + 3x_1x_2^2 - 3x_1x_3^2 - 6x_2x_3^2 - 4x_3^3 .$$

This describes a smooth plane cubic, and we must show that its j-invariant does not vanish. For this, it suffices to verify that the Aronhold S-invariant is non-zero [St, p.173]. The invariant S takes the value 3 on f [St, Prop. 4.4.7].

3.4. *Non-Singular Forms whose Hessians have Seven Singular Points.* We look at a smooth member Y of the family $\left[\begin{smallmatrix} \mathbf{P}_2 & | & 1 \\ \mathbf{P}_2 & | & 1 \end{smallmatrix}\right]$. Let $C_1 = Z(s_1)$ and $C_2 = Z(s_2)$ be two disjoint smooth curves where $s_1, s_2 \in H^0(Y, \mathcal{O}_Y(1,0) \oplus \mathcal{O}_Y(1,1))$ are chosen generically. The blow up X of Y along those curves has the following cup form:

$$3x_1^2x_2 + 3x_1x_2^2 - 3x_1x_3^2 - 6x_2x_3^2 - 4x_3^3 - 3x_1x_4^2 - 6x_2x_4^2 - 4x_4^3 .$$

This polynomial can be written as:

$$\begin{aligned} &-\tfrac{3}{4}x_1^3 - \tfrac{1}{16}(x_1 + 2x_2 + 4x_3)^3 - \tfrac{1}{16}(x_1 + 2x_2 + 4x_4)^3 \\ &+ 3(x_1 + 2x_2)^2\left(\tfrac{1}{4}x_1 - \tfrac{1}{12}(x_1 + 2x_2) + \tfrac{1}{16}(x_1 + 2x_2 + 4x_3) + \tfrac{1}{16}(x_1 + 2x_2 + 4x_4)\right) . \end{aligned}$$

3.5. *Non-Singular Forms whose Hessians have Four Singular Points.* Let Y be again a smooth member of $\left[\begin{smallmatrix} \mathbf{P}_2 & | & 1 \\ \mathbf{P}_2 & | & 1 \end{smallmatrix}\right]$ and choose two disjoint smooth curves $C_1 = Z(s_1)$ and $C_2 = Z(s_2)$, where $s_1, s_2 \in H^0(Y, \mathcal{O}_Y(1,0) \oplus \mathcal{O}_Y(0,1))$ are general sections. The blow up X of Y along those two curves has the polynomial

$$3x_1^2x_2 + 3x_1x_2^2 - 3x_1x_3^2 - 3x_2x_3^2 - 2x_3^3 - 3x_1x_4^2 - 3x_2x_4^2 - 2x_4^3$$

as the cup form, and one checks that its Hessian defines a surface with singularities in the points $[1 : -1 : 0 : 0]$, $[0 : 0 : 0 : 1]$, $[0 : 0 : 1 : 0]$, and $[0 : 0 : -1 : 1]$.

3.6. (A_1). Suppose Y is a smooth member of $\left[\begin{smallmatrix} \mathbf{P}_4 & | & 2 & 1 \\ \mathbf{P}_1 & | & 1 & 1 \end{smallmatrix}\right]$. As we have seen before, Y contains four curves of the type $\{p\} \times \mathbf{P}_1$ with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Define X as the blow up of Y along two of those curves. The cup form of X is then described by the polynomial

$$3x_1^3 + 6x_1^2x_2 - 3x_2x_3^2 - 3x_2x_4^2 + 2x_3^3 + 2x_4^3$$

which defines a surface in $\mathbf{P}_3$ with an A_1-singularity in $[0 : 1 : 0 : 0]$.

3.7. $(2A_1)$. Let X be a smooth member of $\left[\begin{array}{c|ccc} \mathbf{P}_1 & 1 & 1 & 1 \\ \mathbf{P}_1 & 1 & 1 & 1 \\ \mathbf{P}_2 & 1 & 1 & 1 \\ \mathbf{P}_2 & 1 & 1 & 1 \end{array}\right]$. Its cup form is

$$9x_1x_3^2 + 9x_1x_4^2 + 9x_2x_3^2 + 9x_2x_4^2 + 18x_3^2x_4 + 18x_3x_4^2 \\ + 18x_1x_2x_3 + 18x_1x_2x_4 + 36x_1x_3x_4 + 36x_2x_3x_4 .$$

The surface in $\mathbf{P}_3$ defined by this polynomial has two A_1-singularities in $[1:0:0:0]$ and $[0:1:0:0]$.

3.8. $(3A_1)$. Blow up $\mathbf{P}_1 \times \mathbf{P}_1 \times \mathbf{P}_1$ in $C = Z(s)$, where $s \in H^0(\mathcal{O}(1,1,0) \oplus \mathcal{O}(0,1,1))$ is a generically chosen section in order to obtain a manifold X with cup form

$$6x_1x_2x_3 - 3x_1x_4^2 - 3x_2x_4^2 - 3x_3x_4^2 - 4x_4^3 .$$

The corresponding cubic surface has three A_1-singularities in the points $[1:0:0:0]$, $[0:1:0:0]$, and $[0:0:1:0]$.

3.9. $(4A_1)$. Let $V \subset \mathbf{P}_1 \times \mathbf{P}_1 \times \mathbf{P}_1 \times \mathbf{P}_1$ be a smooth divisor in $|\mathcal{O}(a_1, a_2, a_3, a_4)|$, $a_i > 0$, $i = 1, 2, 3, 4$. The cup form of this manifold is described by the polynomial

$$6a_1x_2x_3x_4 + 6a_2x_1x_3x_4 + 6a_3x_1x_2x_4 + 6a_4x_1x_2x_3 ,$$

which defines a surface in $\mathbf{P}_3$ with A_1-singularities in $[1:0:0:0]$, $[0:1:0:0]$, $[0:0:1:0]$, and $[0:0:0:1]$.

3.10. (A_2). Let Y be a smooth complete intersection in the family $\left[\begin{array}{c|cc} \mathbf{P}_4 & 2 & 1 \\ \mathbf{P}_1 & 1 & 1 \end{array}\right]$. This time, blow up Y along a curve of the type $\{p\} \times \mathbf{P}_1$ and in a point. The resulting manifold X has the cup form

$$3x_1^3 + 6x_1^2x_2 - 3x_2x_3^2 + 2x_3^3 + x_4^3 .$$

This polynomial defines a cubic surface with an A_2-singularity in $[0:1:0:0]$.

3.11. $(2A_2)$. Let Y be a projective algebraic threefold with $b_2 = 3$ and cup form $q(x_1, x_2, x_3)$ and suppose that q defines a smooth conic with a transversal line (this happens, e.g., when Y is a $\mathbf{P}_1$-bundle over some surface). The cup form of Y blown up in one point is then

$$q(x_1, x_2, x_3) + x_4^3 .$$

3.12. $(3A_2)$. Let $X = (\mathbf{P}_1 \times \mathbf{P}_1 \times \mathbf{P}_1)^\wedge(p)$ be the blow up of $\mathbf{P}_1 \times \mathbf{P}_1 \times \mathbf{P}_1$ in the point p. The cup form of X is

$$x_4^3 + 6x_1x_2x_3 .$$

3.13. $(2A_1A_2)$. Consider the curve $C = Z(s) \subset \mathbf{P}_1 \times \mathbf{P}_1 \times \mathbf{P}_1$ where $s \in H^0(\mathcal{O}(1,1,0) \oplus \mathcal{O}(0,0,1))$ is a general section, and let X be the blow up of $\mathbf{P}_1 \times \mathbf{P}_1 \times \mathbf{P}_1$ along C. The cup form

$$6x_1x_2x_3 - 3x_1x_4^2 - 3x_2x_4^2 - 2x_4^3$$

of X defines a cubic surface with A_1-singularities in $[1:0:0:0]$ and $[0:1:0:0]$ and an A_2-singularity in $[0:0:1:0]$.

3.14. (D_4^{II}). Let $X := \widehat{\mathbf{P}_1 \times \mathbf{P}}_2(p_1, p_2)$ be the blow up of $\mathbf{P}_1 \times \mathbf{P}_2$ in the points p_1 and p_2. Its cup form is described by the polynomial

$$3x_1x_2^2 + x_3^3 + x_4^3 .$$

This polynomial is the equation of a cubic surface with a D_4-singularity in $[1:0:0:0]$.

3.15. *A Non-Singular Quadric with a Transversal Plane.* Manifolds with such cup forms may be obtained as suitable $\mathbf{P}_1$-bundles over surfaces. Indeed, let Y be a smooth surface with $b_2 = 3$. W. r. t. a suitable basis (h_1, h_2, h_3) of $H^2(Y, \mathbf{Z})$, its cup form is given by $x_1^2 + x_2^2 + x_3^2$. Now, let E be a vector bundle of rank 2 such that $c_1^2(E) - c_2(E) \neq 0$. Let $X := \mathbf{P}(E) \xrightarrow{\pi} Y$ and choose $(\pi^*h_1, \pi^*h_2, \pi^*h_3, c_1(\mathcal{O}_X(1)))$ as a basis of $H^2(X, \mathbf{Z})$. Then, by [OV], Prop. 15, the cup form of X is given by

$$(c_1^2(E) - c_2(E))x_4^3 + x_4(x_1^2 + x_2^2 + x_3^2) .$$

3.16. *A Quadric Cone with a Transversal Plane.* Let Y be a simply connected surface with $b_2 = 3$ and torsion free homology. The cup form of Y is given by a quadratic polynomial $q(x_1, x_2, x_3)$ defining a smooth conic. Thus, the cup form of $Y \times \mathbf{P}_1$ is given by

$$x_4\, q(x_1, x_2, x_3) .$$

4. Real Cubic Forms which are not Cup Forms of Projective Algebraic Manifolds

In the paper [Sch2], the author investigated the restrictions on the real cubic forms of projective manifolds imposed by the so called Hodge-Riemann bilinear relations:

THEOREM 8. *Let X be a Kählerian threefold and $h \in H^2(X, \mathbf{R})$ be a Kähler class. Then the map*

$$\begin{array}{rccc} \langle\,.\,,.\,\rangle : & H^2(X,\mathbf{R}) \times H^2(X,\mathbf{R}) & \longrightarrow & \mathbf{R} \\ & (a,b) & \longmapsto & a \cup b \cup h \end{array}$$

is a non-degenerate, symmetric bilinear form of signature $(2h^{2,0}+1, h^{1,1}-1)$.

One can restate this theorem in such a form as to obtain – at least in theory – some explicit inequalities in the coefficients of cubic polynomials which are satisfied by the cup forms of Kählerian and hence projective algebraic threefolds. The main result of [Sch2] is

THEOREM 9. *For $n \geq 4$, the polynomial*

$$x_0 \left(\frac{4-n}{4} x_0^2 - 3x_1^2 - \cdots - 3x_n^2 \right)$$

cannot occur as the (real) cup form of a projective algebraic threefold with $b_1 = 0$ and $b_3 = 0$.

As a corollary, one obtains the following generalization of a result of Campana and Peternell [CP]:

THEOREM 10. *For $n \geq 4$, twistor spaces over $\sharp_{i=1}^n \mathbf{P}_2$ are not homeomorphic to projective algebraic threefolds.*

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