4. Minkowski vertices and Chebyshev Systems

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identified with the space of trajectories of the geodesic flow ξ. Let λ be the contact form in the space of cooriented contact elements associated with the Hamiltonian function H (see Theorem 1.1). Then the 2-form $d\lambda$ descends to C; this is the symplectic form in question.

The family of Minkowski normals to γ is a curve $\sigma \subset C$. Let $\sigma_0 \subset C$ be the curve that consists of oriented lines through a fixed point x in the plane.

LEMMA 3.3. *The ω-area of the region in C between the curves σ and σ_0 equals zero.*

Proof. Denote by $\widetilde{\gamma}_0$ the set of cooriented contact elements with the foot point at x. Then $\widetilde{\gamma}_0$ is a Legendrian curve. The projections of $\widetilde{\gamma}$ and $\widetilde{\gamma}_0$ along the trajectories of ξ are the curves σ and σ_0. The area under consideration is the integral of the form $d\lambda$ over a film spanned by $\widetilde{\gamma}$ and $\widetilde{\gamma}_0$. By Stokes' theorem, this area equals

$$\int_{\widetilde{\gamma}} \lambda - \int_{\widetilde{\gamma}_0} \lambda = 0$$

since both curves are Legendrian.

In particular, the curves σ and σ_0 intersect at least twice. Therefore at least two Minkowski normals to γ pass through an arbitrary point x in the plane. If the Minkowski metric is associated with a parametrized curve $\gamma(t)$ then the corresponding values of t are the critical points of the function $\left[\gamma(t) - x, \gamma'(t)\right]$.

REMARK. In the Euclidean case a convex closed curve has at least 2 double normals (chords, perpendicular to the curve at both ends). This is still true in the Minkowski setting, provided the indicatrix is centrally symmetric, but does not seem to hold in general.

4. MINKOWSKI VERTICES AND CHEBYSHEV SYSTEMS

This section contains proofs of the 4-vertex theorem in the Minkowski setting (different from the one in [T1]) and a generalization of Theorem 0.1. The arguments used are, more or less, classical; recently the approach via Sturm theory attracted new interest (see [A 1, A 4, A 5, G-M-O]).

Let J have the same meaning as in the previous section and let $J(t)$ be some parameterization of this curve, $0 \leq t \leq T$. Let $\gamma(t)$ be a strictly convex closed smooth curve, parametrized so that the tangent vector $\gamma'(t)$ has the same direction as $J'(t)$ for all t. Denote by $r(t)$ the Minkowski curvature radius at point $\gamma(t)$ and by $k(t) = 1/r(t)$ the Minkowski curvature. Fix a linear coordinate system in the plane, and let $(\gamma_1(t), \gamma_2(t))$ be the coordinates of the point $\gamma(t)$.

LEMMA 4.1. *The function $k'(t)$ is L_2-orthogonal to the functions $\{1, \gamma_1(t), \gamma_2(t)\}$ on the circle $\mathbf{R}/T\mathbf{Z}$.*

Proof. Clearly, $\int_0^T k'(t)dt = 0$. A curve, homothetic to $J(t)$ with the coefficient $r(t)$, is second order tangent to $\gamma(t)$. Therefore $\gamma'(t) = r(t)J'(t)$. One has:

$$\int_0^T k'(t)\gamma(t)dt = -\int_0^T k(t)\gamma'(t)dt = -\int_0^T J'(t)dt = 0\,.$$

Thus $k'(t)$ is orthogonal to $\gamma_1(t)$ and $\gamma_2(t)$.

In the case of a parametrized curve $\gamma(t)$ this means that the function

$$\left(\frac{[\gamma''(t), \gamma'''(t)]}{[\gamma'(t), \gamma''(t)]^2}\right)'$$

is orthogonal to $\{1, \gamma_1(t), \gamma_2(t)\}$ which follows from the easily verified identity:

$$\frac{[\gamma''(t), \gamma'''(t)]}{[\gamma'(t), \gamma''(t)]^2}\gamma'(t) = -\left(\frac{\gamma''(t)}{[\gamma'(t), \gamma''(t)]}\right)'.$$

This orthogonality holds in the case when the function $[\gamma''(t), \gamma'''(t)]$ has zeroes as well.

DEFINITION. A $(2n+1)$-dimensional space of functions on the circle is called a *Chebyshev system* if every function from this space has at most $2n$ zeroes, multiplicities counted.

The functions $\{1, \gamma_1(t), \gamma_2(t)\}$ constitute a Chebyshev system: zeroes of a function $a + b\gamma_1(t) + c\gamma_2(t)$ are the intersections of the line $a + bx + cy$ with the curve γ, and γ is strictly convex. Since Minkowski vertices of γ are critical points of its Minkowski curvature, the 4-vertex theorem follows from the next result found in [Gu 1, A 4, G-M-O].

THEOREM 4.2. *A function f, orthogonal to a $(2n+1)$-dimensional Chebyshev system on the circle, has at least $2n+2$ distinct zeroes.*

Sketch of proof. Assume f has $2n$ simple roots $x_1, \dots, x_{2n}$. There exists a function g in the Chebyshev system with zeroes at $x_1, \dots, x_{2n}$. By definition of Chebyshev systems, this function has no other zeroes. Thus the constant sign intervals of f and g coincide, and $\int fg \neq 0$, a contradiction. (The argument adapts to the general case of fewer and, possibly, multiple roots.)

Theorem 4.2, along with Lemma 3.2, implies the following stronger version of Theorem 0.1.

THEOREM 4.3. *Given a generic parametrized closed strictly convex plane curve $\gamma(t)$, the envelope of the one-parameter family of the lines, generated by the acceleration vectors $\gamma''(t)$, has at least 4 cusps.*

Next, we construct the support function of the curve $\gamma(t)$. Let O be the origin in $\mathbf{R}^2$. The tangent lines to γ at point $\gamma(t)$ and to J at point $J(t)$ are parallel; let $p(t)$ be the coefficient of the homothety with the center at O that takes the latter line to the former.

DEFINITION. The periodic function $p(t)$ is called the *support function* of the curve $\gamma(t)$.

A curve is uniquely determined by its support function. In the Euclidean case the support function is the signed distance from the origin to the oriented tangent lines to γ.

Let $S(t)$ be the parametrized figuratrix, considered as a curve in $\mathbf{R}^2$, and let $\left(S_1(t), S_2(t)\right)$ be its linear coordinates. Consider the collection of curves obtained from $J(t)$ by parallel translations and dilations with positive coefficients.

LEMMA 4.4. *The support functions of these curves are the functions $\{a + bS_1(t) + cS_2(t)\}$ where a, b, c are constants and $a > 0$.*

Proof. Clearly, the support function of $J(t)$ is 1. By Lemma 2.2, $\left[J(t), S(t)\right] = 1$ and $\left[J'(t), S(t)\right] = 0$. Thus the linear functional $[\quad, S(t)]$ equals 1 on the tangent line to the curve J at point $J(t)$. It follows that the support function $p(t)$ of a curve $\gamma(t)$ equals $\left[\gamma(t), S(t)\right]$. Applying the dilation with coefficient a and the parallel translation through vector v to the curve $J(t)$ one obtains the support function

$$p(t) = \left[aJ(t) + v, S(t)\right] = a + \left[v, S(t)\right].$$

The result follows.

The curve $S(t)$ being strictly convex, the vectors $S'(t)$ and $S''(t)$ are everywhere linearly independent. Thus

$$S'''(t) = u(t)\ S''(t) + v(t)\ S'(t)$$

for some T-periodic functions $u(t), v(t)$. Consider the linear differential operator on the circle $\mathbf{R}/T\mathbf{Z}$:

$$L = (d/dt)^3 - u(t)\ (d/dt)^2 - v(t)\ d/dt\,.$$

The kernel of L consists of the functions $a + b\ S_1(t) + c\ S_2(t)$. It follows from the strict convexity of S that these functions constitute a Chebyshev system.

EXAMPLE. If the parameterization $S(t)$ is an affine one then the operator L equals $(d/dt)^3 + k(t)\ (d/dt)$ where $k(t)$ is the affine curvature.

DEFINITION. A linear differential operator of odd degree is called *disconjugate* on the circle $\mathbf{R}/T\mathbf{Z}$ if every function in its kernel is T-periodic and this kernel is a Chebyshev system.

The operator L is disconjugate. In the Euclidean case $S(t)$ is the unit circle, and $L = (d/dt)^3 + d/dt$. Disconjugate operators enjoy the following property proved in [A4, G-M-O].

THEOREM 4.5. *Let L be a disconjugate differential operator on the circle of degree $2n+1$. For every smooth function f on the circle the function $L(f)$ has at least $2n+2$ distinct zeroes.*

Vertices of a curve $\gamma(t)$ present themselves as follows in terms of the support function $p(t)$.

LEMMA 4.6. *A point $\gamma(t_0)$ is a Minkowski vertex if and only if* $L(p)\,(t_0) = 0$.

Proof. Let $p_0(t)$ be the support function of the osculating indicatrix at point $\gamma(t_0)$. Then $(j^2p)(t_0) = (j^2p_0)(t_0)$. If $\gamma(t_0)$ is a vertex then the 3-jets are equal: $(j^3p)(t_0) = (j^3p_0)(t_0)$. Since $L(p_0) = 0$, one has: $L(p)(t_0) = 0$.

Conversely, if $L(p)(t_0) = 0$ then

$$p'''(t_0) = u(t_0)\ p''(t_0) + v(t_0)\ p'(t_0)\,,$$

and the function p_0 satisfies the same equation. Since the 2-jets of p and p_0 at t_0 coincide, it follows that $p'''(t_0) = p_0'''(t_0)$ as well. Therefore the osculating indicatrix is third order tangent to γ at point $\gamma(t_0)$.

Thus Theorem 4.5 again implies the Minkowski 4-vertex theorem.

REMARK. The following result is also known (see the literature cited): if a convex closed curve intersects a curve, homothetic to J, at $2n$ points then it has at least $2n$ Minkowski vertices.

5. CONSERVATIVE TRANSVERSE LINE FIELDS

In this section we discuss the following problem: given a smooth strictly convex closed plane curve γ and a smooth transverse line field l along it, when does a parameterization $\gamma(t)$ exist such that the line $l(t)$ at point $\gamma(t)$ is generated by the acceleration vector $\gamma''(t)$ for all t?

DEFINITION. A transverse line field along a closed plane curve, generated by the acceleration vectors for some parameterization of the curve, is called *conservative*.

Clearly, not every line field is conservative: consider, for example, a field of lines that everywhere make an acute angle with the curve. Theorem 0.1 provides a necessary condition: the envelope of the lines from a conservative line field has at least 4 cusps. Lemma 3.2 gives another one: there exist at least 2 tangent lines to this envelope through every point in the plane.

We start with the following situation. Let M^3 be a contact manifold and let $\widetilde{\gamma} \subset M$ be a closed smooth Legendrian curve. Recall that the characteristic line field η of a contact form λ is the field $Ker\ d\lambda$. Assume that the contact distribution along $\widetilde{\gamma}$ is coorientable; then it can be determined by a contact form. Let η be a line field along $\widetilde{\gamma}$, transverse to the contact distribution.

QUESTION. When does a contact form exist in a vicinity of $\widetilde{\gamma}$ for which η is the characteristic field?

When this is the case we call the field η *characteristic*.

Let λ be some contact form near $\widetilde{\gamma}$ and let v be a vector field along $\widetilde{\gamma}$ that generates the line field η. Consider the 1-form $(i_v\ d\lambda)/\lambda(v)$ and set

$$\beta(\widetilde{\gamma}, \eta) = \int_{\widetilde{\gamma}} \frac{i_v\ d\lambda}{\lambda(v)}\,.$$

THEOREM 5.1. *The number $\beta(\widetilde{\gamma}, \eta)$ does not depend on the choice of the contact form λ nor the vector field v. This number vanishes if and only if the field η is characteristic.*