Introduction

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NONMINIMAL RATIONAL CURVES ON K3 SURFACES

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INTRODUCTION

The following assertion was made, in 1943, by B. Segre ([Se]):

(**S**) *The general quartic surface F contains a finite number $c_h > 0$ of unicursal (i.e., rational) curves of degree $4h$ (for $h = 1, 2, 3, \ldots$).*

This was prompted by the opposite claim made by W. Fr. Meyer at the turn of the century ([Me], §3, pp. 1545–47):

(**M**) *On a generic (quartic surface) F_4 there can lie no (rational curve) R_m $(m = 1, 2, \ldots)$.*

The notation R_m was commonly used to mean: *a rational curve of degree m*, but it is not clear whether Meyer intended to limit his statement to *smooth* rational curves. In fact, the argument he gives in support of his claim makes reasonable sense for smooth curves: it takes $4m + 1$ conditions to express that a quartic surface contains a smooth rational curve of degree m, but these curves depend on $4m$ constants only. Indeed, a parametrization $\varphi \colon \mathbf{P}^1 \to \mathbf{P}^3$ defined by four homogeneous polynomials of degree m depends on $4(m+1)$ coefficients, which are arbitrary up to multiplication by a common scalar; and the ∞^3 automorphisms of $\mathbf{P}^1$ preserve the image of such a map.

The independence of the conditions so expressed would need to be thoroughly examined. But, with this interpretation, assertion (**M**) does hold and can be derived from a celebrated theorem of Max Noether, which is very well established (see [De], thm. 1.2): a generic quartic surface (in a specific sense) has no other divisors than its hypersurface sections; and the arithmetic genus of such divisors is never zero.

Nevertheless, as Segre noticed, Meyer's statement is certainly false when singularities are allowed. Indeed, it is well-known that a general quartic surface in $\mathbf{P}^3$ has 3200 tritangent planes. Each of them meets the surface in a quartic with 3 double points, which has geometric genus zero.

It is interesting to compare with a similar statement proved in 1979 by Mumford and by Bogomolov (see [M-M]):

THEOREM (Bogomolov & Mumford). *Every algebraic K3 surface over* $\mathbf{C}$ *contains a singular rational curve and a pencil of singular elliptic curves.*

Actually, they proved that a complete linear system of curves of *minimal* arithmetic genus (greater than one) on the surface contains at least one irreducible rational member. For a smooth quartic in $\mathbf{P}^3$, which is a special case of K3 surface, this deals with the relatively uninteresting case where $h = 1$.

Assertion $(\mathbf{S})$ would be easy to establish if we knew that *every* complete linear system of curves of arithmetic genus greater than one on a K3 surface has an irreducible rational member. The main innovation of this paper is that, for a restricted family of K3 surfaces, we show the existence of singular rational irreducible members in some complete linear systems which are not minimal.

More precisely, we give a proof of Segre's assertion $(\mathbf{S})$ for $h = 2$ and $h = 3$ (Theorem 3.4). In §4 we establish a similar result for K3 surfaces in $\mathbf{P}^4$ (Theorem 4.1). However, for reasons explained before Lemma 2.5, we have not been able to prove assertion $(\mathbf{S})$ for $h > 3$.

SCHOLIA. What happens in reality is somewhat surprising. Sometimes the problem seems to be very easy, and sometimes very hard. We shall try to explain here why this is so.

First we recall that the set of space curves of a given degree can be viewed as a variety, by a construction usually attributed to Chow, though much of the idea goes back to Cayley ([Ca]). (See [Sh] for a brief, but enlightening discussion, and [H-P] for an almost exhaustive treatment.)

In a few words, let $V \subset \mathbf{P}^N$ be a projective variety of dimension n. We denote by $\widetilde{\mathbf{P}^N}$ the dual space of $\mathbf{P}^N$ and consider the set $\Delta \subset (\widetilde{\mathbf{P}^N})^{n+1} \times V$ of all points $(h_0, \dots, h_n, x)$ such that every hyperplane h_i contains x. One can prove (cf. [Sh], Chap. 1, §6) that Δ is a closed set whose projection in $(\widetilde{\mathbf{P}^N})^{n+1}$ is defined by a single equation F_V. The form F_V is called the *Cayley form* associated with V, and its coefficients are the coordinates of the

Chow point of V. If – instead of looking merely at varieties – one considers all cycles of a given degree m and dimension n, one shows that the Chow points form a projective algebraic variety. This is called the *Chow variety* of all cycles in $\mathbf{P}^N$ of degree m and dimension n.

SCHOLIUM 1. It is known that the Chow variety of space curves of degree m has an irreducible component $\mathcal{R}_m$, of dimension $4m$, whose general element is the Chow point of a smooth rational curve (*cf.* [Co2], Lemma 2.4). Moreover, any irreducible space curve with degree m and geometric genus zero belongs to it.

We denote by $\mathcal{F}_K$ the projective space (of dimension 34) parametrizing all quartic surfaces in $\mathbf{P}^3$. For each value of m, we can consider the incidence correspondence $\mathcal{I}_m \subset \mathcal{R}_m \times \mathcal{F}_K$ consisting of all pairs (γ, F) with $\langle\gamma\rangle \subset F$, where $\langle\gamma\rangle$ denotes the support of the 1-cycle whose Chow point is $\gamma \in \mathcal{R}_m$.

Now a smooth quartic curve Γ of genus 0 in $\mathbf{P}^3$ is contained in a unique quadric surface. From this it is easy to compute that it is contained in precisely ∞^{17} surfaces $F \in \mathcal{F}_K$. (The 17 conditions coming from the Bézout theorem are independent; indeed, given any set of 16 points on Γ, there is a union of two quadrics through them which does not contain Γ.) Thus the incidence correspondence $\mathcal{I}_4$ has dimension $16 + 17 = 33$ above some nonempty open subset of $\mathcal{R}_4$.

But if we look at the family of plane quartics in $\mathbf{P}^3$, we see that its dimension is $14 + 3 = 17$ (one more than the dimension of $\mathcal{R}_4$!). It can be shown that those having 3 double points form a family of dimension $(14 - 3) + 3 = 14$. Now, for a quartic surface to contain such a singular curve Γ, it is enough to impose 11 simple points and the 3 double points, since this also represents $11 + 2 \cdot 3 = 4 \cdot 4 + 1$ intersections. Thus Γ is contained in $\infty^{\geq 20}$ surfaces $F \in \mathcal{F}_K$. It follows that the incidence correspondence $\mathcal{I}_4$ has a component of dimension $\geq 14 + 20 = 34$ above the singular plane quartics. [1])

Hence not only is $\mathcal{I}_4$ reducible, but it has a component of larger dimension than its dimension over the generic point of the Chow variety $\mathcal{R}_4$!

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[1]) In fact, equality holds. The referee suggests the following argument: as a complete intersection, any plane quartic is arithmetically Cohen-Macaulay of arithmetic genus 3. Therefore it imposes exactly $h^0(\mathcal{O}_\Gamma(4)) = 14$ conditions on quartic surfaces.

There is also a family of rational curves of degree 4 and arithmetic genus 1, namely the rational reduced and irreducible quartic curves which are complete intersections of two quadric surfaces (and have a double point). These curves are contained in ∞^{18} surfaces of degree 4.

As a matter of fact, this family of curves $\mathcal{S} \subset \mathcal{R}_4$ has dimension 15. So, the fibres above $\mathcal{S}$ span a 33-dimensional variety, which is also a component of $\mathcal{I}_4$.

This explains why we can say that both Segre and Meyer were right, in some sense: they referred to the images in $\mathcal{F}_K$ of *different components of* $\mathcal{I}_m$.

We proceed with an informal discussion of the case $h = 2$, for which one can also get a pretty clear picture:

SCHOLIUM 2. We refer to Max Noether ([No], § 17) for a discussion of space curves of degree 8. Noether uses several criteria[2]) to establish that a *general* smooth rational octic Γ is contained in no more than two independent quartic surfaces[3]) F_4. Thus the incidence correspondence $\mathcal{I}_8$ has dimension $32+1 = 33$ above some nonempty open subset of $\mathcal{R}_8$ and could not possibly map surjectively onto $\mathcal{F}_K$ if it were irreducible. This is in agreement with Meyer's assertion for degree 8.

However, the complete intersections of a quartic and a quadric have the right dimension (33, which is one more than $\dim \mathcal{R}_8$). Our task will consist in showing that those with 9 double points form a subfamily of $\mathcal{R}_8$ of dimension $33-9 = 24$. Again, since these curves are contained in ∞^{10} quartic surfaces, we have to do with a component of larger dimension than the one above the general point of $\mathcal{R}_8$. We will then show that this component maps onto a dense constructible subset of $\mathcal{F}_K$.

Here is yet another heuristic way to understand why the dimension is one more than normally expected: *In* $\mathcal{I}_8$ *one can obtain a curve* Γ *with* 9 *double points by imposing only* 8 *nodes.*

Indeed, any quadric passing through the 8 nodes and one more point of Γ has $2\cdot 8+1 = 17$ intersections with Γ. Hence Γ is contained in an irreducible quadric Q_0.

Since we are moving in $\mathcal{I}_8 \subset \mathcal{R}_8 \times \mathcal{F}_K$, the divisor Γ is of type $(4,4)$ on Q_0, hence of arithmetic genus 9. But Γ is rational and the assigned singularities are ordinary double points. Hence Γ automatically acquires a ninth singular point.

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[2]) For instance, on a smooth quartic F containing Γ, any other quartic surface F' through Γ cuts out a residual curve Γ'. The linear system $|\Gamma'|$ has dimension 0. Indeed, $p_a(\Gamma) = 0 \Rightarrow (\Gamma)^2 = -2$; whence $(\Gamma')^2 = (\mathcal{O}(4) - \Gamma)^2 = -2$, so that Γ' is an isolated divisor. Hence any other quartic through Γ belongs to the pencil generated by F and F'.

[3]) One cannot leave out the word 'general'. Indeed one also finds some smooth rational curves of degree 8 on any smooth quadric, where they correspond to the divisors of type $(1,7)$. These curves are therefore contained in ∞^9 (reducible) surfaces of degree 4.

As a matter of fact, this family of curves $\mathcal{S} \subset \mathcal{R}_8$ has dimension 24. So the fibres above $\mathcal{S}$ span a 33-dimensional variety, which is also a component of $\mathcal{I}_8$.