3. Rational curves on quartics in P^3

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$\mathcal{W}_Y$ parametrizing all rational, reduced and irreducible curves belonging to $|X_\lambda|$ and having only nodes for singularities. We define $\mathcal{W}_\lambda$ as the union of these varieties $\mathcal{W}_Y$. $\square$

REMARKS. 1) The schemes parametrizing all curves of a given geometric genus, in a linear system on a rational surface, have been well examined (see [Ta], [Ha]). The new feature in Lemma 2.6 is that we "pull them up" to subschemes of the Chow variety $\mathcal{R}_m$.

2) We can think of a smooth cubic surface as being $\mathbf{P}^2$ with six points blown up. Then, if we consider the effect of blowing-down on the curves of the linear system $|X_\lambda|$, we see that Lemma 2.5 has the following interesting consequence: in the system of plane curves of degree 3λ with six λ-fold points, there are some rational, reduced and irreducible curves with only nodes as further singularities.

3. RATIONAL CURVES ON QUARTICS IN $\mathbf{P}^3$

A rational space curve of degree 8 is given as the image of a map $\varphi\colon \mathbf{P}^1 \to \mathbf{P}^3$, defined by four homogeneous polynomials of degree 8. Such maps depend on $4 \cdot 9 = 36$ arbitrary coefficients; hence they are parametrized by $\mathbf{P}^{35}$. Those maps which are generically injective and for which $\varphi(\mathbf{P}^1)$ is a curve of degree 8 correspond to an open subset $U \subset \mathbf{P}^{35}$. By $\varphi \in U$ we mean that the coefficients of φ are in U.

Such a curve Γ is contained in at most one quadric Q. So, it will be convenient to consider the pair (Γ, Q) instead of Γ alone. For simplicity, we shall restrict to the case where Q is smooth. We denote by $\mathcal{L}_Q$ (resp., $\mathcal{L}_C$ and $\mathcal{L}_K$) the quasi-projective variety of smooth quadrics (resp., cubics and quartics) in $\mathbf{P}^3$.

LEMMA 3.1. *The following correspondences between quasi-projective varieties are algebraic and define closed subvarieties:*

a) the incidence correspondence $\mathcal{G} \subset \mathcal{R}_8 \times \mathcal{L}_Q$ parametrizing the rational curves of degree 8 on smooth quadrics;

b) the incidence correspondence $\mathcal{F} \subset \mathcal{R}_8 \times \mathcal{L}_Q \times \mathcal{L}_K$ parametrizing the rational, reduced and irreducible curves of type $(4,4)$ on smooth quadrics which are cut out by smooth quartic surfaces;

c) the incidence correspondence $\mathcal{H} \subset \mathcal{R}_8 \times \mathcal{L}_Q \times \mathcal{F}_K$ parametrizing the rational, reduced and irreducible curves of type $(4,4)$ on smooth quadrics.

Proof. Let

$$F = \left\{ (\varphi, x, Q, K) \in U \times \mathbf{P}^3 \times \mathcal{L}_Q \times \mathcal{L}_K \mid x \in \varphi(\mathbf{P}^1) \subset Q \cap K \right\}.$$

This is a closed subset of $U \times \mathbf{P}^3 \times \mathcal{L}_Q \times \mathcal{L}_K$. Indeed, $\varphi(\mathbf{P}^1) \subset Q \cap K$ means that $\varphi(u : t) \in Q \cap K$ for every $(u : t) \in \mathbf{P}^1$. So, the coefficients of all monomials in $(u : t)$ of $f_Q\big(\varphi(u : t)\big)$ and $f_K\big(\varphi(u : t)\big)$ must vanish (here f_Q and f_K denote the polynomial equations of Q and K). This yields algebraic relations between x and the coefficients of φ, f_Q, and f_K.

Let F_i be any irreducible component of F. Call U_i its first projection. By [H-P] (Chap. 11, §6, Thm. II), there exists an irreducible correspondence $\mathcal{F}_i$ between $\mathcal{L}_Q \times \mathcal{L}_K$ and an irreducible subvariety $\mathcal{R}_{U_i}$ of $\mathcal{R}_8$ which defines the same curves as U_i. We define $\mathcal{F}$ to be the union of the $\mathcal{F}_i$.

Similarly, we define $G = \left\{ (\varphi, x, Q) \in U \times \mathbf{P}^3 \times \mathcal{L}_Q \mid x \in \varphi(\mathbf{P}^1) \subset Q \right\}$. Further, let $H = \left\{ \big(\varphi, x, Q, K\big) \in U \times \mathbf{P}^3 \times \mathcal{L}_Q \times \mathcal{F}_K \mid x \in \varphi(\mathbf{P}^1) \subset Q \cap K \right\}$. As before, G and H are closed subsets of $U \times \mathbf{P}^3 \times \mathcal{L}_Q$, respectively $U \times \mathbf{P}^3 \times \mathcal{L}_Q \times \mathcal{F}_K$. Making use of irreducible components G_i of G, respectively H_i of H, we find irreducible correspondences $\mathcal{G}_i$ and $\mathcal{H}_i$ between $\mathcal{L}_Q$, respectively $\mathcal{L}_Q \times \mathcal{F}_K$ and irreducible varieties $\mathcal{R}_{W_i} \subset \mathcal{R}_8$, respectively $\mathcal{R}_{V_i} \subset \mathcal{R}_8$. Again, we define $\mathcal{G}$ and $\mathcal{H}$ as the unions of these irreducible components.

Finally, we note that $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$, as closed subsets of quasi-projective varieties, are quasi-projective. □

LEMMA 3.2. *The incidence correspondence $\mathcal{F}$ is a quasi-projective variety of dimension* 34.

Proof. Define $\pi_0 : \mathcal{G} \to \mathcal{L}_Q$ by $\pi_0(\gamma, Q) = Q$ (for $\gamma \in \mathcal{R}_8$) and similarly

$$\pi_1 : \mathcal{H} \to \mathcal{G} \text{ by } \pi_1(\gamma, Q, K) = (\gamma, Q) \quad \text{and} \quad \pi : \mathcal{F} \to \mathcal{G} \text{ by } \pi(\gamma, Q, K) = (\gamma, Q).$$

Now, $\mathcal{H}$ is a closed subset of a quasi-projective variety, and $\mathcal{F}_K$ is projective. It follows from [Sh] (Chap. 1, §5, Thm. 3) that $\pi_1(\mathcal{H})$ is closed in $\mathcal{G}$, and hence also quasi-projective.

Since a smooth quadric Q is projectively normal and a linear equivalence class on Q is determined by the bidegree, every reduced and irreducible curve Γ of bidegree $(4, 4)$ is cut out, on Q, by (at least) one irreducible quartic K. Let γ be the Chow point of Γ, so that $(\gamma, Q, K) \in \mathcal{H}$. Then the fibre of π_1 above (γ, Q) contains K and all the reducible quartics through Q. Hence it is of dimension 10.

We also know from Lemma 2.2 that a general member of the linear system of all quartics through Γ is smooth. Hence $\pi_1(\mathcal{H}) = \pi(\mathcal{F})$ and the fibres of π over $\pi(\mathcal{F})$ are also 10-dimensional.

Finally, by Lemma 2.4, $\pi_0 \circ \pi_1$ maps onto $\mathcal{L}_Q$ and all the fibres of $\pi_0|_{\pi(\mathcal{F})}$ have dimension 15. As $\mathcal{L}_Q$ is irreducible of dimension 9, any irreducible component of $\pi(\mathcal{F})$ of maximal dimension has dimension 24. Further, the fibre of π over $\pi(\mathcal{F})$ is 10-dimensional, so $\mathcal{F}$ has dimension 34. □

LEMMA 3.3. *The incidence correspondence $\mathcal{J} \subset \mathcal{R}_{12} \times \mathcal{L}_C \times \mathcal{L}_K$ parametrizing the rational, reduced and irreducible curves which are the complete intersection of a smooth quartic and a smooth cubic surface, is a quasi-projective variety of dimension* 34.

Proof. The argument is much the same as for degree 8. For instance, a curve Γ of degree 12 cannot be contained in more than one cubic C. So, it is convenient to consider the pair (Γ, C) instead of Γ alone. For simplicity, we restrict to the case where C is smooth. Then Lemma 2.6 replaces Lemma 2.4, and the proof of Lemma 3.2 has to be modified mainly for the actual computation of dimensions, which is as follows.

The quasi-projective variety $\mathcal{L}_C$ of smooth cubic surfaces has dimension 19. And the dimension of the family of rational curves $\Gamma \in |X_4|$ is equal to 11, by[5]) Lemma 2.6. Finally, a curve $\Gamma \subset C$ lying on an irreducible quartic K, is also contained in all the reducible quartics through C. Hence the linear system of all quartics through Γ has dimension 4, and a general member is smooth.

Putting everything together, we find $19 + 11 + 4 = 34$ for the dimension of the incidence correspondence $\mathcal{J}$. □

We now come to the proof of assertion (**S**) for $h = 2$ and $h = 3$.

THEOREM 3.4. *The smooth quartics in* $\mathbf{P}^3$ *carrying rational, reduced and irreducible curves of degree* 8, *respectively* 12, *obtained as intersections with smooth quadrics, respectively cubics, form a constructible set of dimension* 34 *in* $\mathcal{L}_K$. *A general quartic in this set carries also some rational curves of degree* 8 *(resp.,* 12*) having only nodes for singularities.*

Proof. Let $p\colon \mathcal{F} \to \mathcal{L}_K$ be the projection map defined by $p(\gamma, Q, K) = K$. Then every fibre of p is finite. Indeed, let us consider the first projection

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[5]) Strictly speaking, this is only a lower bound, since Lemma 2.6 does not count the curves having other singularities than nodes. But the proof of Theorem 3.4 shows that one has in fact equality.

$q\colon \mathcal{F} \to \mathcal{R}_8$. We know from Lemma 3.1 that $\mathcal{F}$ is algebraic. Hence, for any $K \in \mathcal{L}_K$, the push-forward $q_* p^{-1}(K)$ describes an algebraic system of rational curves on K, which cannot be of dimension ≥ 1, by Lemma 1.1, since $K \in \mathcal{L}_K$ is a K3 surface. Hence this algebraic system is finite. Moreover, $\mathcal{F}$ parametrizes only reduced curves of degree 8, which therefore do not belong to more than one quadric. Hence each Chow point γ corresponds to a unique pair (γ, Q). Thus the fibre $p^{-1}(K)$ contains only finitely many points.

Let $E \subset \mathcal{F}$ be an irreducible component of top dimension 34. By the theorem on the dimension of the fibres (see [Sh], Chap. 1, §6, Thm. 7), we see that $\dim p(E) = \dim E = 34$.

On applying Chevalley's theorem to the quasi-projective varieties $\mathcal{F}$ and $\mathcal{L}_K$ and to the finite-type morphism p, we also see that $p(\mathcal{F})$ is constructible, i.e., a finite disjoint union of locally closed subsets V_i. Since $\mathcal{L}_K$ is quasi-projective, so are the V_i.

The same argument works for curves of degree 12, with the map $p\colon \mathcal{J} \to \mathcal{L}_K$. Note that one gets, for a component $E \subset \mathcal{J}$ of maximal dimension,

$$34 \geq \dim p(E) = \dim E \geq 34 .$$

Hence we obtain the same equality as before. $\square$

REMARKS. 1) As expected, singular points other than nodes do not affect the dimensions of the relevant schemes. This is because, roughly speaking, nodes impose the lowest number of conditions for decreasing the geometric genus. However, as is shown by Lemma 2.1, not all curves in Theorem 3.4 have only nodes for singularities.

2) In the proof of Theorem 3.4, we could replace E by its closure $\bar{E}$ in $\mathcal{R}_8 \times \mathcal{F}_Q \times \mathcal{L}_K$, where $\mathcal{F}_Q$ denotes the space of all quadrics in $\mathbf{P}^3$. Now, $\mathcal{R}_8 \times \mathcal{F}_Q$ is a projective variety. Hence $p(\bar{E})$ is closed (cf. [Sh], Chap. 1, §5, Thm. 3) and $p(\bar{E}) = \mathcal{L}_K$. This would account in particular for the rational octics that lie on a quadratic cone, instead of a smooth quadric surface.

4. Rational curves on K3 surfaces in $\mathbf{P}^4$

Let $S_{2,3}$ be a K3 surface spanning $\mathbf{P}^4$ (i.e., not contained in any hyperplane). The notation refers to the fact that such a surface is a smooth complete intersection of a quadric and a cubic threefold. We also write $\mathcal{S}_{2,3}$ for the 43-dimensional quasi-projective variety of all $S_{2,3}$'s in $\mathbf{P}^4$ (see Lemma 4.2). In the present section we prove: