4. Rational curves on K3 surfaces in P^4

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$q\colon \mathcal{F} \to \mathcal{R}_8$. We know from Lemma 3.1 that $\mathcal{F}$ is algebraic. Hence, for any $K \in \mathcal{L}_K$, the push-forward $q_* p^{-1}(K)$ describes an algebraic system of rational curves on K, which cannot be of dimension ≥ 1, by Lemma 1.1, since $K \in \mathcal{L}_K$ is a K3 surface. Hence this algebraic system is finite. Moreover, $\mathcal{F}$ parametrizes only reduced curves of degree 8, which therefore do not belong to more than one quadric. Hence each Chow point γ corresponds to a unique pair (γ, Q). Thus the fibre $p^{-1}(K)$ contains only finitely many points.

Let $E \subset \mathcal{F}$ be an irreducible component of top dimension 34. By the theorem on the dimension of the fibres (see [Sh], Chap. 1, §6, Thm. 7), we see that $\dim p(E) = \dim E = 34$.

On applying Chevalley's theorem to the quasi-projective varieties $\mathcal{F}$ and $\mathcal{L}_K$ and to the finite-type morphism p, we also see that $p(\mathcal{F})$ is constructible, i.e., a finite disjoint union of locally closed subsets V_i. Since $\mathcal{L}_K$ is quasi-projective, so are the V_i.

The same argument works for curves of degree 12, with the map $p\colon \mathcal{J} \to \mathcal{L}_K$. Note that one gets, for a component $E \subset \mathcal{J}$ of maximal dimension,

$$34 \geq \dim p(E) = \dim E \geq 34\,.$$

Hence we obtain the same equality as before. $\square$

REMARKS. 1) As expected, singular points other than nodes do not affect the dimensions of the relevant schemes. This is because, roughly speaking, nodes impose the lowest number of conditions for decreasing the geometric genus. However, as is shown by Lemma 2.1, not all curves in Theorem 3.4 have only nodes for singularities.

2) In the proof of Theorem 3.4, we could replace E by its closure $\bar{E}$ in $\mathcal{R}_8 \times \mathcal{F}_Q \times \mathcal{L}_K$, where $\mathcal{F}_Q$ denotes the space of all quadrics in $\mathbf{P}^3$. Now, $\mathcal{R}_8 \times \mathcal{F}_Q$ is a projective variety. Hence $p(\bar{E})$ is closed (cf. [Sh], Chap. 1, §5, Thm. 3) and $p(\bar{E}) = \mathcal{L}_K$. This would account in particular for the rational octics that lie on a quadratic cone, instead of a smooth quadric surface.

4. RATIONAL CURVES ON K3 SURFACES IN $\mathbf{P}^4$

Let $S_{2,3}$ be a K3 surface spanning $\mathbf{P}^4$ (i.e., not contained in any hyperplane). The notation refers to the fact that such a surface is a smooth complete intersection of a quadric and a cubic threefold. We also write $\mathcal{S}_{2,3}$ for the 43-dimensional quasi-projective variety of all $S_{2,3}$'s in $\mathbf{P}^4$ (see Lemma 4.2). In the present section we prove:

THEOREM 4.1. *The surfaces in $\mathcal{S}_{2,3}$ carrying rational integral curves of degree* 12*, obtained as intersections with smooth quadrics, form a constructible set of dimension* 43 *in* $\mathcal{S}_{2,3}$.

The idea is to consider the curves at issue as belonging to the intersection of two quadrics in $\mathbf{P}^4$. This is a Del Pezzo surface (i.e., its anticanonical sheaf is ample). Hence it is not very different from a cubic surface. In particular, it is rational and we can apply again the results of Tannenbaum.

We write $\mathcal{P}_4$ for the quasi-projective variety of all smooth intersections of two quadrics in $\mathbf{P}^4$. Thus, $\mathcal{P}_4$ is an open subset of the Grassmann variety of pencils of quadrics in $\mathbf{P}^4$.

LEMMA 4.2. *$\mathcal{P}_4$ has dimension* 26*; and $\mathcal{S}_{2,3}$ has dimension* 43.

Proof. The dimension of $\mathcal{P}_4$ is the dimension of the Grassmann variety of rank 2 subspaces of the space of quadratic forms in 5 variables, to wit, $2(15-2)=26$.

Similarly, $\mathcal{S}_{2,3}$ is an open subset of the projective bundle over the space $\mathbf{P}^{14}$ of quadrics with fibre the projectivization of the space of cubic forms modulo (linear) multiples of a quadric. Thus the fibre has dimension $\binom{3+4}{4}-5-1=29$. $\square$

LEMMA 4.3. *Any smooth intersection of two quadrics $P \subset \mathbf{P}^4$ carries (for any positive integer λ) a rational, reduced and irreducible curve Γ_λ of degree $m = 4\lambda$ having only nodes for singularities, which belongs to the linear system $|X_\lambda|$ cut out by all hypersurfaces of degree λ.*

Such curves are parametrized by an irreducible quasi-projective scheme of dimension $m-1$.

Proof. We simply note that $P \in \mathcal{P}_4$ is embedded in $\mathbf{P}^4$ by its anticanonical sheaf. Hence we can apply [Co1], Lemma 1, and the proofs of Lemma 2.5 and Lemma 2.6 carry over with minimal changes. $\square$

In the present paper we are especially interested in the case where $\lambda = 3$. The lemma shows that there exist rational, integral curves of degree $m = 12$ on some surfaces $S \in \mathcal{S}_{2,3}$. They are obtained as intersections with smooth quadrics and have only nodes for singularities.

Proof of Theorem 4.1. Let $\mathcal{G}_{12}$ be the Chow variety of rational curves of degree 12 in $\mathbf{P}^4$. As explained at the beginning of §3, we can work over an open set of reduced and irreducible curves. This will be implied whenever we write a new correspondence. (For simplicity we shall use the same notation, Γ, for a curve and for its Chow point.)

We denote by $\mathcal{L}_Q$ (resp., $\mathcal{L}_C$) the quasi-projective varieties of smooth quadric (resp., cubic) threefolds in $\mathbf{P}^4$. As in Lemma 3.1, the incidence correspondences we are working with can be "pulled up" to define the following algebraic correspondences:

$$\mathcal{H} = \{(\Gamma, P) \in \mathcal{G}_{12} \times \mathcal{P}_4 \mid \Gamma = P \cap C \text{ for some } C \in \mathcal{L}_C \}$$

and

$$\mathcal{J} = \{(\Gamma, S) \in \mathcal{G}_{12} \times \mathcal{S}_{2.3} \mid \Gamma = S \cap Q \text{ for some } Q \in \mathcal{L}_Q \}.$$

In view of Lemmas 4.2 and 4.3, the dimension of $\mathcal{H}$ is equal to $26+(m-1) = 37$. This is also the dimension of its image in $\mathcal{G}_{12}$. Indeed, the fibres of the second projection are finite, since a curve $\Gamma \in \mathcal{G}_{12}$ cannot belong to more than one intersection of two quadrics. (In fact, there is even a map that goes directly from $\mathcal{J}$ to $\mathcal{H}$, but we can do without it.)

To compute the dimension of $\mathcal{J}$, we notice that $\mathcal{H}$ and $\mathcal{J}$ have the same image in $\mathcal{G}_{12}$. Now, a general element in the image of $\mathcal{H}$ corresponds to a curve Γ of degree 12 with 13 distinct nodes and belongs to a pencil of quadrics. But a surface $S \in \mathcal{S}_{2.3}$ is contained in a unique quadric. Hence an element in the fibre of $\mathcal{J}$ above Γ determines a quadric, say Q, in the ∞^1-system of quadrics through Γ and is then determined by the family of all cubic threefolds containing Γ, provided we discount the reducible elements that contain Q.

On the other hand, no more than 24 conditions are required for a cubic hypersurface to contain Γ. In fact it is enough to impose 11 simple points and the 13 double points, since this represents $2 \cdot 13 + 11 = 3 \cdot 12 + 1$ intersections. Therefore, as a vector space, the family of cubics containing Γ has dimension $(\geq)\ 35 - 24 = 11$.

After discounting, as in Lemma 4.2, the 5-dimensional vector space of reducible cubics containing Q as a component, we are left with an ∞^5-system of surfaces[6]) $S \in \mathcal{S}_{2.3}$ containing Γ and contained in Q. As Q varies in a pencil, the fibre of $\mathcal{J}$ above Γ has dimension $(\geq)\ 5 + 1 = 6$.

It follows that $\mathcal{J}$ is of dimension $(\geq)\ 37 + 6 = 43$. Now, let p be the projection map from $\mathcal{J}$ to $\mathcal{S}_{2.3}$. By Lemma 1.1 all the fibres of this map are finite. Since the dimensions are right, as is shown by Lemma 4.2, we conclude exactly as in the proof of Theorem 3.4. $\square$

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[6]) The smoothness can be proved by an extension of Lemma 2.2, in which we replace the divisors in $\mathbf{P}^3$ by divisors in Q.

REMARK. Theorems 3.4 and 4.1, together with [C-S], Example 1.3, clearly imply the following statements:

THEOREM 3.4′ *The smooth quartics in* $\mathbf{P}^3$ *carrying reduced and irreducible curves of degree* 8, *respectively* 12, *and geometric genus* $9 - \delta$ $(0 \leq \delta \leq 9)$, *respectively* $19 - \delta$ $(0 \leq \delta \leq 19)$, *obtained as intersections with smooth quadrics, respectively cubics, and having* δ *nodes, form a constructible set of dimension* 34 *in* $\mathcal{L}_K$.

THEOREM 4.1′ *The surfaces in* $\mathcal{S}_{2,3}$ *carrying integral curves of degree* 12 *and geometric genus* $13 - \delta$ $(0 \leq \delta \leq 13)$, *obtained as intersections with smooth quadrics, form a constructible set of dimension* 43 *in* $\mathcal{S}_{2,3}$.

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