written, it is assumed to be the integers. There are two forms of duality available which we will use. First, Poincaré duality asserts that cap product with the fundamental class $[X] \in H_4(X)$ gives an isomorphism $D : H^2(X) \to H_2(X)$. There is a similar isomorphism when we use $\mathbf{Z}_2$ coefficients which we will also denote by D. For coefficient group $\mathbf{Z}_2$ there is an isomorphism $H : H^2(X;\mathbf{Z}_2) \to \mathrm{Hom}_{\mathbf{Z}_2}\big(H_2(X;\mathbf{Z}_2),\mathbf{Z}_2\big)$ with image the dual space of the vector space $H_2(X;\mathbf{Z}_2)$ over the field $\mathbf{Z}_2$. A basis $b_1,\dots,b_n$ of a finite dimensional vector space V determines an isomorphism between V and its dual V^* by sending b_i to the homomorphism B_i which sends b_i to 1 and b_j to 0 for $j \neq i$. The elements B_i and b_i are said to be *Hom duals*. This isomorphism depends on a choice of basis. However, if we are given any elements $b \in V$, $B \in V^*$, with $B(b) = 1$, then we can always extend $b = b_1$ to a basis of V so that b is the Hom dual of B — just extend b to any basis and then subtract off appropriate multiples of b to get B to evaluate 0 on the other basis elements. The composition of the isomorphism H and the isomorphism determined by the basis gives an isomorphism

$$\overline{H} : H^2(X;\mathbf{Z}_2) \simeq \mathrm{Hom}_{\mathbf{Z}_2}\big(H_2(X;\mathbf{Z}_2),\mathbf{Z}_2\big) \simeq H_2(X;\mathbf{Z}_2)$$

which will be called *Hom duality*. We will call $x \in H_2(X;\mathbf{Z}_2)$ a *Hom dual* of $h \in H^2(X;\mathbf{Z}_2)$ if $H(h)(x) = 1$ since we can always choose a basis of $H_2(X;\mathbf{Z}_2)$ so that $\overline{H}(h) = x$.

We next explore briefly the notions of an even intersection form, spin structure, and spin^c structure for a compact, oriented smooth 4-manifold X. For more details see [B, p. 366–378], [K, p. 20–26, 33–37], [A, p. 95–101], [M, p. 20–25]. The intersection form $H_2(X) \times H_2(X) \to \mathbf{Z}$ is defined by using the intersection product $a \cdot b$ of two homology classes. If the homology classes are represented by smoothly embedded oriented surfaces A, B (i.e. the inclusion maps induce $(i_A)_*[A] = a, (i_B)_*[B] = b$), then $a \cdot b$ may be computed by perturbing A, B up to isotopy to be transversely embedded and summing up the intersections with signs ± 1 according to whether the orientation framing of A followed by the orientation framing of B agrees or disagrees with the orientation framing of X [B, p. 375]. It is always the case that a 2-dimensional homology class in an oriented 4-manifold may be represented by an embedded surface [K, p. 20]. The product $a \cdot b$ may also be computed using Poincaré duality as $a \cdot b = \alpha \cup \beta[X] = \alpha(b)$, where $D\alpha = a, D\beta = b$. There are similar formulas with $\mathbf{Z}_2$ coefficients. A two dimensional $\mathbf{Z}_2$ homology class is not always represented by an embedded oriented surface, but it always may be represented by an embedded nonorientable surface [G, p. 165–166], and there is a similar interpretation of the intersection form in terms of counting geometric transverse intersections. The map $H_2(X) \to H_2(X;\mathbf{Z}_2)$ is surjective exactly when every $\mathbf{Z}_2$ homology class can be represented by an orientable surface.

The universal coefficient sequences with integral and $\mathbf{Z}_2$ coefficients lead to the following diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ext}(H_1(X),\mathbf{Z}) & \longrightarrow & H^2(X;\mathbf{Z}) & \xrightarrow{h_1} & \mathrm{Hom}(H_2(X),\mathbf{Z}) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \rho_2} & & \downarrow \\ 0 & \longrightarrow & \mathrm{Ext}(H_1(X),\mathbf{Z}_2) & \longrightarrow & H^2(X;\mathbf{Z}_2) & \xrightarrow{h_2} & \mathrm{Hom}(H_2(X),\mathbf{Z}_2) & \longrightarrow & 0 \end{array}$$

The homomorphisms h_1 and h_2 are related to the intersection form:

$$h_1(\alpha)(b) = a \cdot b, \quad h_2(\alpha)(b) = a \cdot b \mod 2$$

where $D(\alpha) = a$ with either $\mathbf{Z}$ or $\mathbf{Z}_2$ coefficients. The homomorphisms ρ_i come from reduction mod 2. The intersection form is called *even* if $x \cdot x$ is an even number for all $x \in H_2(X)$. An integral class $a \in H_2(X)$ so that $a \cdot x = x \cdot x \mod 2$ for all x is called *characteristic* for the intersection form. a is characteristic if the homomorphism $S(x) = x \cdot x \mod 2$ is the image of a under the homomorphism $k\colon H_2(X;\mathbf{Z}) \longrightarrow \mathrm{Hom}\big(H_2(X),\mathbf{Z}_2\big)$ where $k(a)(x) = a \cdot x \mod 2$. If a is a characteristic class, and α is its Poincaré dual, then $h_1(\alpha)(x) = a \cdot x = x \cdot x \mod 2$. Thus a is characteristic iff its Poincaré dual α satisfies $h_2\rho_1(\alpha) = \rho_2 h_1(\alpha) = S$. Since the form is even iff $S = 0$, this means that the form is even iff for a characteristic, $D\alpha = a$, then $h_2\big(\rho_1(\alpha)\big) = 0$.

The existence of characteristic classes uses the nondegeneracy of the intersection form and Poincaré duality with $\mathbf{Z}_2$ coefficients. The intersection pairing $H_2(X;\mathbf{Z}) \times H_2(X;\mathbf{Z}) \longrightarrow \mathbf{Z}$ factors through $\Gamma \times \Gamma \longrightarrow \mathbf{Z}$ where $\Gamma = H_2(X;\mathbf{Z})/\mathrm{Tors}$, and when we reduce mod 2, through $\Gamma_2 \times \Gamma_2 \longrightarrow \mathbf{Z}_2$ where $\Gamma_2 = \Gamma \otimes \mathbf{Z}_2$. The existence follows from $\Gamma \longrightarrow \Gamma_2$ being surjective and $\Gamma_2 \longrightarrow \mathrm{Hom}(\Gamma_2,\mathbf{Z}_2)$ being an isomorphism. For this last isomorphism, note both sides are $\mathbf{Z}_2$-vector spaces and have dimension equal to $\mathrm{rank}\, H_2(X;\mathbf{Z})$. The isomorphism is established once the map is seen to be injective. This follows from the fact that the intersection form is nondegenerate due to Poincaré duality: for each $v, \exists\, w$ with $w \cdot v = 1$; in fact, $w = D\psi$ where ψ is the Hom dual of v:

$$w \cdot v = D\psi \cdot v = H(\psi)(v) = 1\,.$$

The second Stiefel-Whitney class $w_2(X) \in H^2(X;\mathbf{Z}_2)$ belongs to a family of characteristic classes. A good reference for properties of the Stiefel-Whitney classes and characteristic classes in general is [MS]. For our discussion here we need to know three of its properties. First, it is related to the characteristic classes discussed above in that its Poincaré dual $D\big(w_2(X)\big)$ satisfies the characteristic property for the $\mathbf{Z}_2$ intersection form:

$$H\big(w_2(X)\big)\,(z) = D\big(w_2(X)\big)\cdot z = z\cdot z$$

for all $z \in H_2(X;\mathbf{Z}_2)$. When we restrict to the image of integral classes, we get the statement that $h_2\big(w_2(X)\big)\,(x) = x\cdot x \mod 2$. This means that if $D(\alpha_1)$ is an integral characteristic class, then $h_2\big(w_2 - \rho_1(\alpha_1)\big) = 0$. The second property that $w_2(X)$ satisfies is that an oriented manifold X has a spin structure iff $w_2(X) = 0$. A spin structure on X is a lifting of the structure group of the tangent bundle of X from $SO(4)$ to its universal (double)cover $spin(4)$. The third property which $w_2(X)$ possesses relates to spin^c structures. The group $spin^c(4)$ is the double cover $spin(4)\times S^1/\pm 1$ of $SO(4)\times S^1$ induced from the double cover on each factor. A spin^c structure on X consists of a lifting of the structure group of the product of the tangent bundle of X and a chosen line bundle L over X from $SO(4)\times S^1$ to $spin^c(4)$. The 4-manifold X has a spin^c structure exactly when the second Stiefel-Whitney class $w_2(X) = \rho_1(\alpha)$ for some integral class α ([HH, p. 169], [M, p. 25]).

We now give the argument why $w_2(X)$ always lifts to an integral class from the excellent expository account of Seiberg-Witten invariants by S. Akbulut [A, p. 95]. We saw above that the existence of an integral characteristic class means there is an integral class α_1 so that $h_2\big(w_2(X) - \rho_1(\alpha_1)\big) = 0$. Hence $w_2 - \rho_1(\alpha_1)$ comes from $\mathrm{Ext}\big(H_1(X),\mathbf{Z}_2\big)$. But the map $\mathrm{Ext}\big(H_1(X),\mathbf{Z}\big) \longrightarrow \mathrm{Ext}\big(H_1(X),\mathbf{Z}_2\big)$ is surjective since the first group gives the torsion subgroup of $H_1(X)$ and the latter the 2-torsion subgroup. Hence $\exists\,\alpha_2 \in \mathrm{Ext}\big(H_1(X),\mathbf{Z}\big) \hookrightarrow H^2(X;\mathbf{Z})$ with $\rho_1(\alpha_2) = w_2 - \rho_1(\alpha_1)$. This implies $w_2 = \rho_1(\alpha_1 + \alpha_2)$ is the image of an integral cohomology class. Note that this also means that the Poincaré dual $D(w_2)$ is the image of an integral homology class.

With this background, we return now to our initial example M. To see that $w_2(M) \neq 0$, Habegger [H] notes that if $\mathbf{RP}^2 = \{[(x,x)]\}$ is the image of the diagonal $\triangle$ in $S^2\times S^2$ under the quotient, then $[\triangle]\cdot[\triangle] = 2$ in $S^2\times S^2$ leads to $[\mathbf{RP}^2]\cdot[\mathbf{RP}^2] = 1$ in M. If $[\mathbf{RP}^2] = D\gamma$, where $\gamma\in H^2(M;\mathbf{Z}_2)$, then we have $(\gamma\cup\gamma)\,[M] = [\mathbf{RP}^2]\cdot[\mathbf{RP}^2] = 1$. Thus $w_2(M)\cup\gamma = \gamma\cup\gamma \neq 0$, which implies $w_2(M)\neq 0$ and thus M is not spin.

Next note $\pi_1(M) = \mathbf{Z}_2 = H_1(M)$ since M is double covered by $S^2\times S^2$. Using this and the computation of Euler characteristic as $\chi(M) = \chi(S^2\times S^2)/2 = 2$, Habegger shows $\operatorname{rank} H_2(M) = 0$. Evenness of the intersection form follows. The universal coefficient sequences for M are:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{Z}_2 & \xrightarrow{\simeq} & \mathbf{Z}_2 & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow{\simeq} & & \downarrow{\rho} & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathbf{Z}_2 & \longrightarrow & \mathbf{Z}_2 \oplus \mathbf{Z}_2 & \longrightarrow & \mathbf{Z}_2 & \longrightarrow & 0
\end{array}$$

Consider the homology class Dw_2. We claim that it is represented by the embedded sphere which is the image under the quotient of $S^2 \times p$ or $p \times S^2$ in $S^2 \times S^2$. Here p is a chosen point in S^2, say (1,0,0). To see this, note that $(S^2 \times p) \cap \triangle = (p,p)$ and the intersection is transverse. This gives us $[S^2 \times p]_2 \cdot [\mathbf{RP}^2] = 1$ in M, and $[S^2 \times p]_2$ is therefore a nonzero class in $H_2(M;\mathbf{Z}_2)$ — the subscript 2 indicates that here we are viewing $[S^2 \times p]$ as a $\mathbf{Z}_2$ homology class rather than an integral class. This implies $[S^2 \times p]$ must be nonzero in $H_2(M) \simeq \mathbf{Z}_2$. Its Poincaré dual in $H^2(M) \simeq \mathbf{Z}_2$ must therefore be the unique nonzero class which reduces mod 2 to $w_2(M)$. This is reflected in our commutative diagram. Evenness is reflected through the upper right term being zero, and the image of w_2 to the Hom term being zero. Exactness implies $w_2 \in \mathbf{Z}_2 \oplus \mathbf{Z}_2$ must come from the Ext term. Note that under the isomorphism $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \simeq H^2(M;\mathbf{Z}_2) \simeq \mathrm{Hom}_{\mathbf{Z}_2}\big(H_2(M;\mathbf{Z}_2),\mathbf{Z}_2\big), w_2$ maps to a nonzero homomorphism which evaluates zero on $[S^2 \times p]_2$ and one on $[\mathbf{RP}^2]$.

What is true here is that the class $[\mathbf{RP}^2]$ in $H_2(M;\mathbf{Z}_2)$ does not come from an integral class. The evaluation of w_2 on $[\mathbf{RP}^2]$ and $[S^2 \times p]_2$ distinguishes these classes. Thus, these two surfaces generate $H_2(M;\mathbf{Z}_2) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$ and the intersection form with respect to this basis is just $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. We also note that $[\mathbf{RP}^2]$ cannot be represented by an oriented surface N. If it were, $[N]$ would represent an element of $H_2(M)$, and as we have seen, $[\mathbf{RP}^2]$ is not in the image of the homomorphism $H_2(M) \longrightarrow H_2(M;\mathbf{Z}_2)$ since the form is even.

How typical is this example ? First, if X has an even intersection form and $w_2(X) \neq 0$, then there must be a class $a \in H_2(X;\mathbf{Z}_2)$ with $a \cdot a \neq 0$ detecting $w_2(X) \neq 0$ so that a does not come from an integral class. This class a can be taken as a Hom dual of $w_2(X)$, not the Poincaré dual. In our example, $[\mathbf{RP}^2]$ is the Hom dual to $w_2(M)$ (using the basis $[S^2 \times p]_2, \mathbf{RP}^2$ to form the duality) since $H\big(w_2(M)\big)\big([\mathbf{RP}^2]\big) = 1$ and $H\big(w_2(M)\big)\big([S^2 \times p]_2\big) = 0$. Of course, no such example can have $H_2(X) \longrightarrow H_2(X;\mathbf{Z}_2)$ surjective, which implies X is not simply connected. Secondly, $H_2(X;\mathbf{Z}_2)$ is always represented by embedded surfaces, orientable or nonorientable. All classes in the image