1. Introduction

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **43 (1997)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **21.07.2024**

L'Enseignement Mathématique, t. 43 (1997), p. 33–54

BOTT-CHERN FORMS AND ARITHMETIC INTERSECTIONS

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ABSTRACT. Let $\overline{\mathcal{E}} : 0 \to \bar{S} \to \bar{E} \to \bar{Q} \to 0$ be a short exact sequence of hermitian vector bundles with metrics on S and Q induced from that on E. We compute the Bott-Chern form $\widetilde{\phi}(\overline{\mathcal{E}})$ corresponding to any characteristic class ϕ, assuming $\bar{E}$ is projectively flat. The result is used to obtain a new presentation of the Arakelov Chow ring of the arithmetic Grassmannian.

1. INTRODUCTION

Arakelov theory is an intersection theory for varieties over rings $\mathcal{O}_F$ of algebraic integers, analogous to the usual one over fields. The fundamental idea is that in order to have a good theory of intersection numbers, one has to include information at the infinite primes.

The work of Arakelov in dimension two has been generalized by Gillet and Soulé to higher dimensional *arithmetic varieties* X, by which we mean regular, projective and flat schemes over $\operatorname{Spec}\mathbf{Z}$. They define an *arithmetic Chow ring* $\widehat{CH}(X)_{\mathbf{Q}}$ whose elements are represented by cycles on X together with Green currents on $X(\mathbf{C})$. The theory is a blend of arithmetic, algebraic geometry and complex hermitian geometry. For example, the Faltings height of an arithmetic variety X is realized as an "arithmetic degree" with respect to a hermitian line bundle over X.

A *hermitian vector bundle* $\bar{E} = (E, h)$ over X is an algebraic vector bundle E on X together with a hermitian metric h on the corresponding holomorphic vector bundle $E(\mathbf{C})$ on the complex manifold $X(\mathbf{C})$. To such an object one associates arithmetic Chern classes $\widehat{c}(\bar{E})$ with values in $\widehat{CH}(X)$. These satisfy most of the usual properties of Chern classes, with one exception: given a short exact sequence of hermitian vector bundles

$$(1) \qquad \overline{\mathcal{E}}: \; 0 \to \bar{S} \to \bar{E} \to \bar{Q} \to 0$$

the class $\widehat{c}(\bar{S})\widehat{c}(\bar{Q}) - \widehat{c}(\bar{E})$ vanishes when $\bar{E}$ is the orthogonal direct sum of $\bar{S}$ and $\bar{Q}$. In general however this difference is non-zero and is the image in $\widehat{CH}(X)$ of a differential form on $X(\mathbf{C})$, the *Bott-Chern form* associated to the exact sequence $\overline{\mathcal{E}}$.

These secondary characteristic classes were originally defined by Bott and Chern [BC] with applications to value distribution theory. They later occured in the work of Donaldson [Do] on Hermitian-Einstein metrics. Bismut, Gillet and Soulé [BiGS] gave a new axiomatic definition for Bott-Chern forms, suitable for use in arithmetic intersection theory. Given an exact sequence of hermitian holomorphic vector bundles as in (1), we have $c(\bar{S})c(\bar{Q}) - c(\bar{E}) = dd^c\eta$ for some form η; the Bott Chern form of $\overline{\mathcal{E}}$ is a natural choice of such an η.

Calculating these forms is important because they give relations in the arithmetic Chow ring of an arithmetic variety. No systematic work has appeared on this; rather one finds scattered calculations throughout the literature (see for example [BC], [C1], [D], [GS2], [GSZ], [Ma], [Mo]). We confine ourselves to the case where the metrics on S and Q are induced from the one on E. Our goal is to give explicit formulas for the Bott-Chern forms corresponding to *any* characteristic class, when they can be expressed in terms of the characteristic classes of the bundles involved. This is not always possible as these forms are not closed in general; however the situation is completely understood when E is a projectively flat bundle. The results build on the work of Bott, Chern, Cowen, Deligne, Gillet, Soulé and Maillot. Some of our calculations overlap with previous work, but with simpler proofs.

The main application we give to arithmetic intersection theory is a new presentation of the Arakelov Chow ring of the Grassmannian over Spec $\mathbf{Z}$. Maillot [Ma] gave a presentation of this ring and formulated an "arithmetic Schubert calculus". We hope our work contributes towards a better understanding of these intersections.

This paper is organized as follows. Section 2 is a review of some basic material on invariant and symmetric functions. In §3 we recall the hermitian geometry we will need, including the definition of Bott-Chern forms. The basic tool for calculating these forms is reviewed in §4, with some applications that have appeared before in the literature. §7 is mainly an exposition of the arithmetic intersection theory that we require. The rest of the paper is new. In sections 5 and 6 we derive formulas for computing Bott-Chern forms of short exact sequences (with the induced metrics) for any characteristic class when $\bar{E}$ is flat or more generally projectively flat. We emphasize the central role played by the *power sum forms* in the results; to our knowledge this phenomenon has not been observed before. The combinatorial identities involving harmonic numbers that we encounter are also interesting. Sections 2-6 contain results in hermitian complex geometry and may be read without prior knowledge of Arakelov theory. § 8 applies our calculations to obtain a presentation of the Arakelov Chow ring of the arithmetic Grassmannian.

This should be regarded as a companion paper to [T]; both papers will be part of the author's 1997 University of Chicago thesis. I wish to thank my advisor William Fulton for many useful conversations and exchanges of ideas.

2. Invariant and symmetric functions

The symmetric group S_n acts on the polynomial ring $\mathbf{Z}[x_1, x_2, \ldots, x_n]$ by permuting the variables, and the ring of invariants $\Lambda(n) = \mathbf{Z}[x_1, x_2, \ldots, x_n]^{S_n}$ is the ring of symmetric polynomials. For $B = \mathbf{Q}$ or $\mathbf{C}$, let $\Lambda(n, B) = \Lambda(n) \otimes_{\mathbf{Z}} B$.

Let $e_k(x_1, \ldots, x_n)$ be the k-th elementary symmetric polynomial in the variables $x_1, \ldots, x_n$ and $p_k(x_1, \ldots, x_n) = \sum_i x_i^k$ the k-th power sum. The fundamental theorem on symmetric functions states that $\Lambda(n) = \mathbf{Z}[e_1, \ldots, e_n]$ and that $e_1, \ldots, e_n$ are algebraically independent. For λ a partition, i.e. a decreasing sequence $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m$ of nonnegative integers, define $p_\lambda := \prod_{i=1}^m p_{\lambda_i}$. It is well known that the p_λ's form an additive $\mathbf{Q}$-basis for the ring of symmetric polynomials (cf. [M], § I.2). The two bases are related by Newton's identity:

$$p_k - e_1 p_{k-1} + e_2 p_{k-2} - \cdots + (-1)^k k e_k = 0 . \tag{2}$$

Another important set of symmetric functions related to the cohomology ring of grassmannians are the Schur polynomials. For a partition λ as above, the Schur polynomial s_λ is defined by

$$s_\lambda(x_1, \ldots, x_n) = \frac{1}{\Delta} \cdot \det(x_i^{\lambda_j + n - j})_{1 \leqslant i, j \leqslant n} ,$$

where $\Delta = \prod_{1 \leqslant i < j \leqslant n} (x_i - x_j)$ is the Vandermonde determinant. The s_λ for all λ of length $m \leqslant n$ form a $\mathbf{Z}$-basis of $\Lambda(n)$ (cf. [M], § I.3).

Let $\mathbf{C}[T_{ij}]$ $(1 \leqslant i, j \leqslant n)$ be the coordinate ring of the space $M_n(\mathbf{C})$ of $n \times n$ matrices. $GL_n(\mathbf{C})$ acts on matrices by conjugation, and we let