

3. Hermitian differential geometry

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$I(n) = \mathbf{C}[T_{ij}]^{GL_n(\mathbf{C})}$ denote the corresponding graded ring of invariants. There is an isomorphism $\tau: I(n) \rightarrow \Lambda(n, \mathbf{C})$ given by evaluating an invariant polynomial ϕ on the diagonal matrix $\text{diag}(x_1, \dots, x_n)$. We will often identify ϕ with the symmetric polynomial $\tau(\phi)$. We will need to consider invariant polynomials with rational coefficients; let $I(n, \mathbf{Q}) \simeq \mathbf{Q}[x_1, x_2, \dots, x_n]^{S_n}$ be the corresponding ring.

Given $\phi \in I(n)_k$, let ϕ' be a k -multilinear form on $M_n(\mathbf{C})$ such that

$$\phi'(gA_1g^{-1}, \dots, gA_kg^{-1}) = \phi'(A_1, \dots, A_k)$$

for $g \in GL(n, \mathbf{C})$ and $\phi(A) = \phi'(A, A, \dots, A)$. Such forms are most easily constructed for the power sums p_k by setting

$$p'_k(A_1, A_2, \dots, A_k) = \text{Tr}(A_1A_2 \cdots A_k).$$

For p_λ we can take $p'_\lambda = \prod p'_{\lambda_i}$. Since the p_λ 's are a basis of $\Lambda(n, \mathbf{Q})$, it follows that one can use the above constructions to find multilinear forms ϕ' for any $\phi \in I(n)_k$.

An explicit formula for ϕ' is given by polarizing ϕ :

$$\phi'(A_1, \dots, A_k) = \frac{(-1)^k}{k!} \sum_{j=1}^k \sum_{i_1 < \dots < i_j} (-1)^j \phi(A_{i_1} + \dots + A_{i_j}).$$

Although above formula for ϕ' is symmetric in A_1, \dots, A_k , this property is not needed for the applications that follow.

3. HERMITIAN DIFFERENTIAL GEOMETRY

Let X be a complex manifold, E a rank n holomorphic vector bundle over X . Denote by $A^k(X, E)$ the C^∞ sections of $\Lambda^k T^*X \otimes E$, where T^*X denotes the cotangent bundle of X . In particular $A^k(X)$ is the space of smooth complex k -forms on X . Let $A^{p,q}(X)$ the space of smooth complex forms of type (p, q) on X and $A(X) := \bigoplus_p A^{p,p}(X)$. The decomposition $A^1(X, E) = A^{1,0}(X, E) \oplus A^{0,1}(X, E)$ induces a decomposition $D = D^{1,0} + D^{0,1}$ of each connection D on E . Let $d = \partial + \bar{\partial}$ and $d^c = (\partial - \bar{\partial})/(4\pi i)$.

Assume now that E is equipped with a hermitian metric h . The pair (E, h) is called a *hermitian vector bundle*. The metric h induces a canonical connection $D = D(h)$ such that $D^{0,1} = \bar{\partial}_E$ and D is *unitary*, i.e.

$$dh(s, t) = h(Ds, t) + h(s, Dt), \quad \text{for all } s, t \in A^0(X, E).$$

The connection D is called the *hermitian holomorphic connection* of (E, h) . D can be extended to E -valued forms by using the Leibnitz rule:

$$D(\omega \otimes s) = d\omega \otimes s + (-1)^{\deg \omega} \omega \otimes Ds.$$

The composite

$$K = D^2 : A^0(X, E) \rightarrow A^2(X, E)$$

is $A^0(X)$ -linear; hence $K \in A^2(X, \text{End}(E))$. In fact

$$K = D^{1,1} \in A^{1,1}(X, \text{End}(E)),$$

because $D^{0,2} = \bar{\partial}_E^2 = 0$, so $D^{2,0}$ also vanishes by unitarity. K is called the *curvature* of D .

Given a hermitian vector bundle $\bar{E} = (E, h)$ and an invariant polynomial $\phi \in I(n)$ there is an associated differential form $\phi(\bar{E}) := \phi\left(\frac{i}{2\pi}K\right)$, defined locally by identifying $\text{End}(E)$ with $M_n(\mathbf{C})$; $\phi(\bar{E})$ makes sense globally on X since ϕ is invariant by conjugation. These differential forms are d and d^c closed and have the following properties (cf. [BC]):

- (i) The de Rham cohomology class of $\phi(\bar{E})$ is independent of the metric h and coincides with the usual characteristic class from topology.
- (ii) For every holomorphic map $f : X \rightarrow Y$ of complex manifolds,

$$f^*(\phi(E, h)) = \phi(f^*E, f^*h).$$

One thus obtains the *Chern forms* $c_k(\bar{E})$ with $c_k = e_k(x_1, \dots, x_n)$, the *power sum forms* $p_k(\bar{E})$, the *Chern character form* $ch(\bar{E})$ with $ch(x_1, \dots, x_n) = \sum_i \exp(x_i) = \sum_k \frac{1}{k!} p_k$, etc.

We fix some more notation: A direct sum $\bar{E}_1 \oplus \bar{E}_2$ of hermitian vector bundles will always mean the orthogonal direct sum $(E_1 \oplus E_2, h_1 \oplus h_2)$. Let $\tilde{A}(X)$ be the quotient of $A(X)$ by $\text{Im } \partial + \text{Im } \bar{\partial}$. If ω is a closed form in $A(X)$ the cup product $\wedge \omega : \tilde{A}(X) \rightarrow \tilde{A}(X)$ and the operator $dd^c : \tilde{A}(X) \rightarrow A(X)$ are well defined.

Let $\mathcal{E} : 0 \rightarrow S \rightarrow E \rightarrow Q \rightarrow 0$ be an exact sequence of holomorphic vector bundles on X . Choose arbitrary hermitian metrics h_S, h_E, h_Q on S, E, Q respectively. Let

$$\bar{\mathcal{E}} = (\mathcal{E}, h_S, h_E, h_Q) : 0 \rightarrow \bar{S} \rightarrow \bar{E} \rightarrow \bar{Q} \rightarrow 0.$$

Note that we do not in general assume that the metrics h_S or h_Q are induced from h_E . We say that $\bar{\mathcal{E}}$ is *split* when $(E, h_E) = (S \oplus Q, h_S \oplus h_Q)$ and \mathcal{E} is the obvious exact sequence. Following [GS2], we have the following

THEOREM 1. *Let $\phi \in I(n)$ be any invariant polynomial. There is a unique way to attach to every exact sequence $\bar{\mathcal{E}}$ a form $\tilde{\phi}(\bar{\mathcal{E}})$ in $\tilde{A}(X)$ in such a way that:*

- (i) $dd^c \tilde{\phi}(\bar{\mathcal{E}}) = \phi(\bar{S} \oplus \bar{Q}) - \phi(\bar{E})$;
- (ii) *for every map $f : X \rightarrow Y$ of complex manifolds, $\tilde{\phi}(f^*(\bar{\mathcal{E}})) = f^* \tilde{\phi}(\bar{\mathcal{E}})$;*
- (iii) *if $\bar{\mathcal{E}}$ is split, then $\tilde{\phi}(\bar{\mathcal{E}}) = 0$.*

In [BC], Bott and Chern solved the equation $dd^c \tilde{\phi}(\bar{\mathcal{E}}) = \phi(\bar{S} \oplus \bar{Q}) - \phi(\bar{E})$ when the metrics on S and Q are induced from the metric on E . In [BiGS] a new axiomatic definition of these forms was given, more generally for an acyclic complex of holomorphic vector bundles on X .

The following useful calculation is an immediate consequence of the definition ([GS2], Prop. 1.3.1):

PROPOSITION 1. *Let ϕ and ψ be two invariant polynomials. Then*

$$\widetilde{\phi + \psi}(\bar{\mathcal{E}}) = \tilde{\phi}(\bar{\mathcal{E}}) + \tilde{\psi}(\bar{\mathcal{E}}),$$

and

$$\tilde{\phi}\tilde{\psi}(\bar{\mathcal{E}}) = \tilde{\phi}(\bar{\mathcal{E}})\psi(\bar{E}) + \phi(\bar{S} \oplus \bar{Q})\tilde{\psi}(\bar{\mathcal{E}}) = \tilde{\phi}(\bar{\mathcal{E}})\psi(\bar{S} \oplus \bar{Q}) + \phi(\bar{E})\tilde{\psi}(\bar{\mathcal{E}}).$$

Proof. One checks that the right hand side of these identities satisfies the three properties of Theorem 1 that characterize the left hand side. \square

We will also need to know the behaviour of \tilde{c} when $\bar{\mathcal{E}}$ is twisted by a line bundle. The following is a consequence of [GS2], Prop. 1:3.3:

PROPOSITION 2. *For any hermitian line bundle \bar{L} ,*

$$\tilde{c}_k(\bar{\mathcal{E}} \otimes \bar{L}) = \sum_{i=1}^k \binom{n-i}{k-i} \tilde{c}_i(\bar{\mathcal{E}}) c_1(\bar{L})^{k-i}.$$