$I(n) = \mathbf{C}[T_{ij}]^{GL_n(\mathbf{C})}$ denote the corresponding graded ring of invariants. There is an isomorphism $\tau\colon I(n) \to \Lambda(n, \mathbf{C})$ given by evaluating an invariant polynomial ϕ on the diagonal matrix $\mathrm{diag}(x_1, \ldots, x_n)$. We will often identify ϕ with the the symmetric polynomial $\tau(\phi)$. We will need to consider invariant polynomials with rational coefficients; let $I(n, \mathbf{Q}) \simeq \mathbf{Q}[x_1, x_2, \ldots, x_n]^{S_n}$ be the corresponding ring.

Given $\phi \in I(n)_k$, let ϕ' be a k-multilinear form on $M_n(\mathbf{C})$ such that

$$\phi'(gA_1g^{-1}, \ldots, gA_kg^{-1}) = \phi'(A_1, \ldots, A_k)$$

for $g \in GL(n, \mathbf{C})$ and $\phi(A) = \phi'(A, A, \ldots, A)$. Such forms are most easily constructed for the power sums p_k by setting

$$p'_k(A_1, A_2, \ldots, A_k) = \mathrm{Tr}(A_1 A_2 \cdots A_k)\,.$$

For p_λ we can take $p'_\lambda = \prod p'_{\lambda_i}$. Since the p_λ's are a basis of $\Lambda(n, \mathbf{Q})$, it follows that one can use the above constructions to find multilinear forms ϕ' for any $\phi \in I(n)_k$.

An explicit formula for ϕ' is given by polarizing ϕ:

$$\phi'(A_1 \ldots, A_k) = \frac{(-1)^k}{k!} \sum_{j=1}^{k} \sum_{i_1 < \ldots < i_j} (-1)^j \phi(A_{i_1} + \ldots + A_{i_j})\,.$$

Although above formula for ϕ' is symmetric in $A_1 \ldots, A_k$, this property is not needed for the applications that follow.

3. Hermitian differential geometry

Let X be a complex manifold, E a rank n holomorphic vector bundle over X. Denote by $A^k(X, E)$ the C^∞ sections of $\Lambda^k T^*X \otimes E$, where T^*X denotes the cotangent bundle of X. In particular $A^k(X)$ is the space of smooth complex k-forms on X. Let $A^{p,q}(X)$ the space of smooth complex forms of type (p, q) on X and $A(X) := \bigoplus_p A^{p,p}(X)$. The decomposition $A^1(X, E) = A^{1,0}(X, E) \bigoplus A^{0,1}(X, E)$ induces a decomposition $D = D^{1,0} + D^{0,1}$ of each connection D on E. Let $d = \partial + \overline{\partial}$ and $d^c = (\partial - \overline{\partial})/(4\pi i)$.

Assume now that E is equipped with a hermitian metric h. The pair (E, h) is called a *hermitian vector bundle*. The metric h induces a canonical connection $D = D(h)$ such that $D^{0,1} = \overline{\partial}_E$ and D is *unitary*, i.e.

$$d\,h(s, t) = h(Ds, t) + h(s, Dt), \qquad \text{for all } s, t \in A^0(X, E)\,.$$

The connection D is called the *hermitian holomorphic connection* of (E,h). D can be extended to E-valued forms by using the Leibnitz rule:

$$D(\omega \otimes s) = d\omega \otimes s + (-1)^{\deg \omega} \omega \otimes Ds .$$

The composite

$$K = D^2 : A^0(X,E) \to A^2(X,E)$$

is $A^0(X)$-linear; hence $K \in A^2\big(X, \mathrm{End}(E)\big)$. In fact

$$K = D^{1,1} \in A^{1,1}\big(X, \mathrm{End}(E)\big) ,$$

because $D^{0,2} = \bar{\partial}_E^2 = 0$, so $D^{2,0}$ also vanishes by unitarity. K is called the *curvature* of D.

Given a hermitian vector bundle $\bar{E} = (E,h)$ and an invariant polynomial $\phi \in I(n)$ there is an associated differential form $\phi(\bar{E}) := \phi\big(\frac{i}{2\pi}K\big)$, defined locally by identifying $\mathrm{End}(E)$ with $M_n(\mathbf{C})$; $\phi(\bar{E})$ makes sense globally on X since ϕ is invariant by conjugation. These differential forms are d and d^c closed and have the following properties (cf. [BC]):

(i) The de Rham cohomology class of $\phi(\bar{E})$ is independent of the metric h and coincides with the usual characteristic class from topology.

(ii) For every holomorphic map $f : X \to Y$ of complex manifolds,

$$f^*\big(\phi(E,h)\big) = \phi(f^*E, f^*h) .$$

One thus obtains the *Chern forms* $c_k(\bar{E})$ with $c_k = e_k(x_1, \dots, x_n)$, the *power sum forms* $p_k(\bar{E})$, the *Chern character form* $ch(\bar{E})$ with $ch(x_1, \dots, x_n) = \sum_i \exp(x_i) = \sum_k \frac{1}{k!} p_k$, etc.

We fix some more notation: A direct sum $\bar{E}_1 \bigoplus \bar{E}_2$ of hermitian vector bundles will always mean the orthogonal direct sum $(E_1 \bigoplus E_2, h_1 \oplus h_2)$. Let $\widetilde{A}(X)$ be the quotient of $A(X)$ by $\mathrm{Im}\,\partial + \mathrm{Im}\,\bar{\partial}$. If ω is a closed form in $A(X)$ the cup product $\wedge\omega : \widetilde{A}(X) \to \widetilde{A}(X)$ and the operator $dd^c : \widetilde{A}(X) \to A(X)$ are well defined.

Let $\mathcal{E}$: $0 \to S \to E \to Q \to 0$ be an exact sequence of holomorphic vector bundles on X. Choose arbitrary hermitian metrics h_S, h_E, h_Q on S, E, Q respectively. Let

$$\bar{\mathcal{E}} = (\mathcal{E}, h_S, h_E, h_Q) : \ 0 \to \bar{S} \to \bar{E} \to \bar{Q} \to 0 .$$

Note that we do not in general assume that the metrics h_S or h_Q are induced from h_E. We say that $\bar{\mathcal{E}}$ is *split* when $(E, h_E) = (S \bigoplus Q, h_S \oplus h_Q)$ and $\mathcal{E}$ is the obvious exact sequence. Following [GS2], we have the following

THEOREM 1. *Let $\phi \in I(n)$ be any invariant polynomial. There is a unique way to attach to every exact sequence $\overline{\mathcal{E}}$ a form $\widetilde{\phi}(\overline{\mathcal{E}})$ in $\widetilde{A}(X)$ in such a way that:*

(i) $dd^c\widetilde{\phi}(\overline{\mathcal{E}}) = \phi(\bar{S} \bigoplus \bar{Q}) - \phi(\bar{E})$;

(ii) *for every map $f : X \to Y$ of complex manifolds,* $\widetilde{\phi}\big(f^*(\overline{\mathcal{E}})\big) = f^*\widetilde{\phi}(\overline{\mathcal{E}})$;

(iii) *if $\overline{\mathcal{E}}$ is split, then* $\widetilde{\phi}(\overline{\mathcal{E}}) = 0$.

In [BC], Bott and Chern solved the equation $dd^c\widetilde{\phi}(\overline{\mathcal{E}}) = \phi(\bar{S} \bigoplus \bar{Q}) - \phi(\bar{E})$ when the metrics on S and Q are induced from the metric on E. In [BiGS] a new axiomatic definition of these forms was given, more generally for an acyclic complex of holomorphic vector bundles on X.

The following useful calculation is an immediate consequence of the definition ([GS2], Prop. 1.3.1):

PROPOSITION 1. *Let ϕ and ψ be two invariant polynomials. Then*

$$\widetilde{\phi + \psi}(\overline{\mathcal{E}}) = \widetilde{\phi}(\overline{\mathcal{E}}) + \widetilde{\psi}(\overline{\mathcal{E}}),$$

and

$$\widetilde{\phi\psi}(\overline{\mathcal{E}}) = \widetilde{\phi}(\overline{\mathcal{E}})\psi(\bar{E}) + \phi(\bar{S} \oplus \bar{Q})\widetilde{\psi}(\overline{\mathcal{E}}) = \widetilde{\phi}(\overline{\mathcal{E}})\psi(\bar{S} \oplus \bar{Q}) + \phi(\bar{E})\widetilde{\psi}(\overline{\mathcal{E}}).$$

Proof. One checks that the right hand side of these identities satisfies the three properties of Theorem 1 that characterize the left hand side. □

We will also need to know the behaviour of $\widetilde{c}$ when $\overline{\mathcal{E}}$ is twisted by a line bundle. The following is a consequence of [GS2], Prop. 1:3.3:

PROPOSITION 2. *For any hermitian line bundle $\bar{L}$,*

$$\widetilde{c_k}(\overline{\mathcal{E}} \otimes \bar{L}) = \sum_{i=1}^{k} \binom{n-i}{k-i} \widetilde{c_i}(\overline{\mathcal{E}}) c_1(\bar{L})^{k-i}.$$