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$I(n) = \mathbf{C}[T_{ij}]^{GL_n(\mathbf{C})}$  denote the corresponding graded ring of invariants. There is an isomorphism  $\tau: I(n) \rightarrow \Lambda(n, \mathbf{C})$  given by evaluating an invariant polynomial  $\phi$  on the diagonal matrix  $\text{diag}(x_1, \dots, x_n)$ . We will often identify  $\phi$  with the symmetric polynomial  $\tau(\phi)$ . We will need to consider invariant polynomials with rational coefficients; let  $I(n, \mathbf{Q}) \simeq \mathbf{Q}[x_1, x_2, \dots, x_n]^{S_n}$  be the corresponding ring.

Given  $\phi \in I(n)_k$ , let  $\phi'$  be a  $k$ -multilinear form on  $M_n(\mathbf{C})$  such that

$$\phi'(gA_1g^{-1}, \dots, gA_kg^{-1}) = \phi'(A_1, \dots, A_k)$$

for  $g \in GL(n, \mathbf{C})$  and  $\phi(A) = \phi'(A, A, \dots, A)$ . Such forms are most easily constructed for the power sums  $p_k$  by setting

$$p'_k(A_1, A_2, \dots, A_k) = \text{Tr}(A_1A_2 \cdots A_k).$$

For  $p_\lambda$  we can take  $p'_\lambda = \prod p'_{\lambda_i}$ . Since the  $p_\lambda$ 's are a basis of  $\Lambda(n, \mathbf{Q})$ , it follows that one can use the above constructions to find multilinear forms  $\phi'$  for any  $\phi \in I(n)_k$ .

An explicit formula for  $\phi'$  is given by polarizing  $\phi$ :

$$\phi'(A_1, \dots, A_k) = \frac{(-1)^k}{k!} \sum_{j=1}^k \sum_{i_1 < \dots < i_j} (-1)^j \phi(A_{i_1} + \dots + A_{i_j}).$$

Although above formula for  $\phi'$  is symmetric in  $A_1, \dots, A_k$ , this property is not needed for the applications that follow.

### 3. HERMITIAN DIFFERENTIAL GEOMETRY

Let  $X$  be a complex manifold,  $E$  a rank  $n$  holomorphic vector bundle over  $X$ . Denote by  $A^k(X, E)$  the  $C^\infty$  sections of  $\Lambda^k T^*X \otimes E$ , where  $T^*X$  denotes the cotangent bundle of  $X$ . In particular  $A^k(X)$  is the space of smooth complex  $k$ -forms on  $X$ . Let  $A^{p,q}(X)$  the space of smooth complex forms of type  $(p, q)$  on  $X$  and  $A(X) := \bigoplus_p A^{p,p}(X)$ . The decomposition  $A^1(X, E) = A^{1,0}(X, E) \oplus A^{0,1}(X, E)$  induces a decomposition  $D = D^{1,0} + D^{0,1}$  of each connection  $D$  on  $E$ . Let  $d = \partial + \bar{\partial}$  and  $d^c = (\partial - \bar{\partial})/(4\pi i)$ .

Assume now that  $E$  is equipped with a hermitian metric  $h$ . The pair  $(E, h)$  is called a *hermitian vector bundle*. The metric  $h$  induces a canonical connection  $D = D(h)$  such that  $D^{0,1} = \bar{\partial}_E$  and  $D$  is *unitary*, i.e.

$$dh(s, t) = h(Ds, t) + h(s, Dt), \quad \text{for all } s, t \in A^0(X, E).$$

The connection  $D$  is called the *hermitian holomorphic connection* of  $(E, h)$ .  $D$  can be extended to  $E$ -valued forms by using the Leibnitz rule:

$$D(\omega \otimes s) = d\omega \otimes s + (-1)^{\deg \omega} \omega \otimes Ds.$$

The composite

$$K = D^2 : A^0(X, E) \rightarrow A^2(X, E)$$

is  $A^0(X)$ -linear; hence  $K \in A^2(X, \text{End}(E))$ . In fact

$$K = D^{1,1} \in A^{1,1}(X, \text{End}(E)),$$

because  $D^{0,2} = \bar{\partial}_E^2 = 0$ , so  $D^{2,0}$  also vanishes by unitarity.  $K$  is called the *curvature* of  $D$ .

Given a hermitian vector bundle  $\bar{E} = (E, h)$  and an invariant polynomial  $\phi \in I(n)$  there is an associated differential form  $\phi(\bar{E}) := \phi\left(\frac{i}{2\pi}K\right)$ , defined locally by identifying  $\text{End}(E)$  with  $M_n(\mathbf{C})$ ;  $\phi(\bar{E})$  makes sense globally on  $X$  since  $\phi$  is invariant by conjugation. These differential forms are  $d$  and  $d^c$  closed and have the following properties (cf. [BC]):

- (i) The de Rham cohomology class of  $\phi(\bar{E})$  is independent of the metric  $h$  and coincides with the usual characteristic class from topology.
- (ii) For every holomorphic map  $f : X \rightarrow Y$  of complex manifolds,

$$f^*(\phi(E, h)) = \phi(f^*E, f^*h).$$

One thus obtains the *Chern forms*  $c_k(\bar{E})$  with  $c_k = e_k(x_1, \dots, x_n)$ , the *power sum forms*  $p_k(\bar{E})$ , the *Chern character form*  $ch(\bar{E})$  with  $ch(x_1, \dots, x_n) = \sum_i \exp(x_i) = \sum_k \frac{1}{k!} p_k$ , etc.

We fix some more notation: A direct sum  $\bar{E}_1 \oplus \bar{E}_2$  of hermitian vector bundles will always mean the orthogonal direct sum  $(E_1 \oplus E_2, h_1 \oplus h_2)$ . Let  $\tilde{A}(X)$  be the quotient of  $A(X)$  by  $\text{Im } \partial + \text{Im } \bar{\partial}$ . If  $\omega$  is a closed form in  $A(X)$  the cup product  $\wedge \omega : \tilde{A}(X) \rightarrow \tilde{A}(X)$  and the operator  $dd^c : \tilde{A}(X) \rightarrow A(X)$  are well defined.

Let  $\mathcal{E} : 0 \rightarrow S \rightarrow E \rightarrow Q \rightarrow 0$  be an exact sequence of holomorphic vector bundles on  $X$ . Choose arbitrary hermitian metrics  $h_S, h_E, h_Q$  on  $S, E, Q$  respectively. Let

$$\bar{\mathcal{E}} = (\mathcal{E}, h_S, h_E, h_Q) : 0 \rightarrow \bar{S} \rightarrow \bar{E} \rightarrow \bar{Q} \rightarrow 0.$$

Note that we do not in general assume that the metrics  $h_S$  or  $h_Q$  are induced from  $h_E$ . We say that  $\bar{\mathcal{E}}$  is *split* when  $(E, h_E) = (S \oplus Q, h_S \oplus h_Q)$  and  $\mathcal{E}$  is the obvious exact sequence. Following [GS2], we have the following

THEOREM 1. *Let  $\phi \in I(n)$  be any invariant polynomial. There is a unique way to attach to every exact sequence  $\bar{\mathcal{E}}$  a form  $\tilde{\phi}(\bar{\mathcal{E}})$  in  $\tilde{A}(X)$  in such a way that:*

- (i)  $dd^c \tilde{\phi}(\bar{\mathcal{E}}) = \phi(\bar{S} \oplus \bar{Q}) - \phi(\bar{E})$ ;
- (ii) *for every map  $f : X \rightarrow Y$  of complex manifolds,  $\tilde{\phi}(f^*(\bar{\mathcal{E}})) = f^* \tilde{\phi}(\bar{\mathcal{E}})$ ;*
- (iii) *if  $\bar{\mathcal{E}}$  is split, then  $\tilde{\phi}(\bar{\mathcal{E}}) = 0$ .*

In [BC], Bott and Chern solved the equation  $dd^c \tilde{\phi}(\bar{\mathcal{E}}) = \phi(\bar{S} \oplus \bar{Q}) - \phi(\bar{E})$  when the metrics on  $S$  and  $Q$  are induced from the metric on  $E$ . In [BiGS] a new axiomatic definition of these forms was given, more generally for an acyclic complex of holomorphic vector bundles on  $X$ .

The following useful calculation is an immediate consequence of the definition ([GS2], Prop. 1.3.1):

PROPOSITION 1. *Let  $\phi$  and  $\psi$  be two invariant polynomials. Then*

$$\widetilde{\phi + \psi}(\bar{\mathcal{E}}) = \tilde{\phi}(\bar{\mathcal{E}}) + \tilde{\psi}(\bar{\mathcal{E}}),$$

and

$$\tilde{\phi}\tilde{\psi}(\bar{\mathcal{E}}) = \tilde{\phi}(\bar{\mathcal{E}})\psi(\bar{E}) + \phi(\bar{S} \oplus \bar{Q})\tilde{\psi}(\bar{\mathcal{E}}) = \tilde{\phi}(\bar{\mathcal{E}})\psi(\bar{S} \oplus \bar{Q}) + \phi(\bar{E})\tilde{\psi}(\bar{\mathcal{E}}).$$

*Proof.* One checks that the right hand side of these identities satisfies the three properties of Theorem 1 that characterize the left hand side.  $\square$

We will also need to know the behaviour of  $\tilde{c}$  when  $\bar{\mathcal{E}}$  is twisted by a line bundle. The following is a consequence of [GS2], Prop. 1:3.3:

PROPOSITION 2. *For any hermitian line bundle  $\bar{L}$ ,*

$$\tilde{c}_k(\bar{\mathcal{E}} \otimes \bar{L}) = \sum_{i=1}^k \binom{n-i}{k-i} \tilde{c}_i(\bar{\mathcal{E}}) c_1(\bar{L})^{k-i}.$$