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8. Arakelov Chow rings of Grassmannians

In this section $G = G(r, n)$ will denote the Grassmannian over $\operatorname{Spec} \mathbf{Z}$. Over any field k, G parametrizes the r-dimensional linear subspaces of a vector space over k. Let

$$\overline{\mathcal{E}} : \; 0 \to \bar{S} \to \bar{E} \to \bar{Q} \to 0 \tag{13}$$

denote the universal exact sequence of vector bundles over G. Here the trivial bundle $E(\mathbf{C})$ is given the trivial metric and the tautological subbundle $S(\mathbf{C})$ and quotient bundle $Q(\mathbf{C})$ the induced metrics. The homogeneous space $G(\mathbf{C}) \simeq U(n)/\big(U(r) \times U(n-r)\big)$ is a complex manifold. $G(\mathbf{C})$ is endowed with a natural $U(n)$-invariant metric coming from the Kähler form $\eta_G = c_1\big(\overline{Q(\mathbf{C})}\big)$.

G is a smooth arithmetic scheme and $G(\mathbf{C})$ with the metric coming from η_G is a hermitian symmetric space, so we have an Arakelov Chow ring $CH(\overline{G})$. Note that since the hermitian vector bundles in (13) are invariant under the action of $U(n)$, their Chern forms are harmonic, and thus the arithmetic characteristic classes obtained are all elements of $CH(\overline{G})$. Maillot [Ma] found a presentation of $CH(\overline{G})$, using the above observation and the short exact sequence (11). We wish to offer another description of this ring, based on the calculations in this paper.

First recall the geometric picture: for the ordinary Chow ring we have

$$CH(G) = \frac{\mathbf{Z}\big[c(S), c(Q)\big]}{\langle c(S)c(Q) = 1\rangle}. \tag{14}$$

(see for instance [F], Example 14.6.6). If $x_1, \ldots, x_r$ are the Chern roots of S, $y_1, \ldots, y_s$ are the Chern roots of Q, $H = S_r \times S_{n-r}$ is the product of two symmetric groups, and t is a formal variable, then (14) can be rewritten

$$CH(G) = \frac{\mathbf{Z}[x_1, \ldots, x_r, y_1, \ldots, y_s]^H}{\Big\langle \prod_i (1 + x_i t) \prod_j (1 + y_j t) = 1 \Big\rangle}. \tag{15}$$

Maillot's presentation of $CH(\overline{G})$ is an analogue of (14); ours will be an analogue of (15). We introduce $2n$ variables

$$\widehat{x}_1, \ldots, \widehat{x}_r, \widehat{y}_1, \ldots, \widehat{y}_s, x_1, \ldots, x_r, y_1, \ldots, y_s$$

and consider the rings

$$A = \mathbf{Z}[\widehat{x}_1, \ldots, \widehat{x}_r, \widehat{y}_1, \ldots, \widehat{y}_s]^H \quad \text{and} \quad B = \mathbf{R}[x_1, \ldots, x_r, y_1, \ldots, y_s]^H$$

and the ring homomorphism $\omega : A \to B$ defined by $\omega(\widehat{x}_i) = x_i$ and $\omega(\widehat{y}_j) = y_j$. A ring structure is defined on the abelian group $A \oplus B$ by setting

$$(\widehat{x},x')*(\widehat{y},y')=\left(\widehat{xy},\omega(\widehat{x})y'+x'\omega(\widehat{y})\right).$$

We will adopt the convention that $\widehat{\alpha}$ denotes $(\widehat{\alpha},0)$, β denotes $(0,\beta)$, and any product $\prod x_iy_j$ denotes $(0,\prod x_iy_j)$; the multiplication $*$ is thus characterized by the properties $\widehat{\alpha}*\beta=\alpha\beta$ and $\beta_1*\beta_2=0$. We now define two sets of relations in $(A\oplus B)[t]$:

$$\mathcal{R}_1:\ \prod_i(1+x_it)\prod_j(1+y_jt)=1,$$

$$\mathcal{R}_2:\ \prod_i(1+\widehat{x}_it)*\prod_j(1+\widehat{y}_jt)*\left(1+t\sum_j\frac{\log(1+y_jt)}{1+y_jt}\right)=1,$$

and let $\mathcal{A}$ denote the quotient of the graded ring $A\oplus B$ by these relations. Using this notation we can state

THEOREM 6. *There is a unique ring isomorphism $\Phi:\mathcal{A}\to CH(\overline{G})$ such that*

$$\Phi\Big(\prod_i(1+\widehat{x}_it^i)\Big)=\sum_i\widehat{c}_i(\bar{S})t^i,\qquad \Phi\Big(\prod_j(1+\widehat{y}_jt^j)\Big)=\sum_j\widehat{c}_j(\bar{Q})t^j,$$
$$\Phi\Big(\prod_i(1+x_it^i)\Big)=\sum_i a\big(c_i(\bar{S})\big)t^i,\qquad \Phi\Big(\prod_j(1+y_jt^j)\Big)=\sum_j a\big(c_j(\bar{Q})\big)t^j.$$

Proof. The isomorphism Φ of $\mathcal{A}$ with $CH(\overline{G})$ is obtained exactly as in [Ma], Theorem 4.0.5. The key fact is that since G has a cellular decomposition (in the sense of [F], Ex. 1.9.1), it follows that $CH^{p,p-1}(G)=0$ for all p (using the excision exact sequence for the groups $CH^{*,*}(G)$; cf. [G], §8). Summing the sequence (11) over all p gives

$$0\longrightarrow\mathcal{H}(G_{\mathbf{R}})\xrightarrow{a}CH(\overline{G})\xrightarrow{\zeta}CH(G)\longrightarrow0. \tag{16}$$

We can now use our knowledge of the rings $\mathcal{H}(G_{\mathbf{R}})$ and $CH(G)$ together with the five lemma, as in loc. cit. The multiplication $*$ is a consequence of the general identity $a(x)y=a\big(x\omega(y)\big)$ in $\widehat{CH}(G)$. To complete the argument we must show that the relation $\widehat{c}(\bar{S})\widehat{c}(\bar{Q})=1+a\big(\widetilde{c}(\bar{\mathcal{E}})\big)$ translates to the relation $\mathcal{R}_2$ above.

Let $p_i(y)$ be the i-th power sum in the variables $y_1\dots,y_s$, identified under Φ with the class $a\big(p_i(\bar{Q})\big)$ in $CH(\overline{G})$ (we will use such identifications freely in the sequel). We also define $p_a(t)=\sum_{i=0}^{\infty}(-1)^{i+1}\mathcal{H}_ip_i(y)t^{i+1}$. Proposition 3 implies that

$$\widehat{c}_t(\bar{S})\widehat{c}_t(\bar{Q}) = 1 + a\big(\widetilde{c}_t(\bar{\mathcal{E}})\big) = 1 - p_a(t)\,, \tag{17}$$

where the subscript t denotes the corresponding Chern polynomial. Multiplying both sides of (17) by $1 + p_a(t)$ and using the properties of multiplication in $\mathcal{A}$ gives the equivalent form

$$\widehat{c}_t(\bar{S}) * \widehat{c}_t(\bar{Q}) * \big(1 + p_a(t)\big) = 1\,. \tag{18}$$

We now note that the *harmonic number generating function*

$$\sum_{i=0}^{\infty} \mathcal{H}_i t^i = \frac{t}{1-t} + \frac{1}{2}\frac{t^2}{1-t} + \frac{1}{3}\frac{t^3}{1-t} + \cdots = \frac{\log(1-t)}{t-1}\,.$$

It follows that

$$p_a(-t) = \sum_{i=0}^{\infty} \mathcal{H}_i p_i(y) t^{i+1} = t\sum_{j=1}^{s}\sum_{i=0}^{\infty} \mathcal{H}_i (y_j t)^i = -t\sum_{j=1}^{s} \frac{\log(1-y_j t)}{1-y_j t}$$

and thus

$$p_a(t) = t\sum_{j=1}^{s} \frac{\log(1+y_j t)}{1+y_j t}\,.$$

Substituting this in equation (18) gives relation $\mathcal{R}_2$. □

Theorem 6 shows that the relations in the Arakelov Chow ring of G are the classical geometric ones perturbed by a new "arithmetic factor" of $1 + p_a(t)$. While this factor is closely related to the power sums $p_i(\bar{Q})$, the most natural basis of symmetric functions for doing calculations in $CH(G)$ is the basis of Schur polynomials (corresponding to the Schubert classes; see for example [F], § 14.7). The arithmetic analogues of the special Schubert classes involve the power sum perturbation above; multiplication formulas are thus quite complicated (see [Ma]).

In geometry the Chern roots x_i and y_j all "live" on the complete flag variety above G. There are certainly natural line bundles on the flag variety whose first Chern classes correspond to the roots in Theorem 6. However on flag varieties the situation is more complicated and our knowledge is not as complete. We refer the reader to [T] for more details.

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