THE NORMALISER ACTION AND STRONGLY MODULAR LATTICES

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ABSTRACT. We derive group theoretical methods to test if a lattice is strongly modular. We then apply these methods to the lattices of rational irreducible maximal finite groups.

1. INTRODUCTION

Let $L \subseteq \mathbf{R}^d$ be an even integral lattice in the Euclidean space of dimension d and let $L^\# \subseteq \mathbf{R}^d$ be its dual lattice. Let $\pi(L)$ be the set of all intermediate lattices $L \leq L' \leq L^\#$ that are inverse images of sums of Sylow subgroups of the finite abelian group $L^\#/L$. Then, L is said to be *strongly modular* if L is similar to L' for all $L' \in \pi(L)$ (cf. [Que 96]). Recall that L and L' are called *similar* if there exists $s \in GL(\mathbf{R}L)$ and $a \in \mathbf{R}_{>0}$ such that $Ls = L'$ and $(vs, ws) = a(v, w)$ for all $v, w \in \mathbf{R}L$, where $(\,,)$ denotes the Euclidean scalar product.

The automorphism group

$$G := \mathrm{Aut}(L) = \{g \in O(\mathbf{R}L) \mid Lg \subseteq L\}$$

is conjugate to a finite subgroup of $GL_d(\mathbf{Z})$. Since G acts as group automorphisms on $L^\#/L$ it preserves the lattices $L' \in \pi(L)$.

In Section 3 it is shown that the similarities $L' \to L$ normalise G. So one may use the normaliser

$$N_{GL_d(\mathbf{Q})}(G) := \{n \in GL_d(\mathbf{Q}) \mid n^{-1}gn \in G \quad \text{for all} \quad g \in G\}$$

*) Supported by the DFG.

of G in $GL_d(\mathbf{Q})$ to test strong modularity of L. In the next section we derive some methods for explicitly constructing elements of $N_{GL_d(\mathbf{Q})}(G)$.

Every finite subgroup of $GL_d(\mathbf{Q})$ is a subgroup of the automorphism group of an integral lattice. In particular the maximal finite subgroups of $GL_d(\mathbf{Q})$ are automorphism groups of distinguished lattices. A subgroup of $GL_d(\mathbf{Q})$ is called rational irreducible if it does not preserve a proper subspace $\neq \{0\}$ of $\mathbf{Q}^d$. The rational irreducible maximal finite, abbreviated to *r.i.m.f.*, subgroups of $GL_d(\mathbf{Q})$ are classified for $d < 32$ (cf. [PlN 95], [NeP 95], [Neb 95], [Neb 96], [Neb 96a]). Their invariant lattices provide many examples of strongly modular lattices. The following theorem is proved by applying the methods derived in Section 4.

THEOREM. *In dimension $d < 32$, all even lattices $L \subseteq \mathbf{R}^d$ that are preserved by a r.i.m.f. group and satisfy $L^\#/L \cong (\mathbf{Z}/l\mathbf{Z})^{d/2}$ for some $l \in \mathbf{N}$ are strongly modular, except for the lattices of the r.i.m.f. group* $[\pm \mathrm{Alt}_6 .2^2]_{16}$ *in $GL_{16}(\mathbf{Q})$ (cf.* [NeP 95]*).*

2. Preliminaries and Notation

The main strategy in this paper is the application of the following *normaliser principle*.

Let G be a group acting on a set S, H a subgroup of the group of transformations of S. Then the normaliser of G in H acts on the set of G-orbits.

In our situation $G = \mathrm{Aut}(L)$ is the automorphism group of an integral lattice L in the Euclidean space $\mathbf{R}L \cong \mathbf{R}^d$. By writing the action of G on $\mathbf{R}L$ with respect to a $\mathbf{Z}$-basis $(b_1, \ldots, b_d)$ of L, G becomes a finite subgroup of $GL_d(\mathbf{Z})$. Then $G = \mathrm{Aut}(F) = \{g \in GL_d(\mathbf{Z}) \mid gFg^{tr} = F\}$ where F is the Gram matrix $F = \left((b_i, b_j)\right)_{i,j=1}^d$ of L.

For the rest of this article let $H = GL_d(\mathbf{Q})$, $G \leq H$, be a finite subgroup of H, and let $N := N_H(G)$ be its normaliser. We also assume that G contains the negative unit matrix, $-I_d \in G$.

We apply the normaliser principle to the following three situations.

(i) $S = \{L \subseteq \mathbf{Q}^d \mid L = \sum_{i=1}^d \mathbf{Z}b_i$ for a basis $(b_1, \ldots, b_d)$ of $\mathbf{Q}^d\}$, the set of $\mathbf{Z}$-lattices of rank d in $\mathbf{Q}^d$, and the action of H on S is right multiplication: $S \times H \to S$, $(L, h) \mapsto Lh := \{lh \mid l \in L\}$. Then the set of G-fixed points is

$$\mathcal{Z}(G) := \{L \in S \mid Lg = L \text{ for all } g \in G\},$$

the set of G-invariant lattices.

(ii) $S = \{F \in M_d(\mathbf{Q}) \mid F = F^{tr},\ F$ positive definite$\}$, the set of positive definite symmetric matrices, where x^{tr} denotes the transposed matrix of $x \in M_d(\mathbf{Q})$ and the action of H on S is $S \times H \to S$, $(F,h) \mapsto hFh^{tr}$. Then the set of G-fixed points is

$$\mathcal{F}_{>0}(G) := \{F \in S \mid gFg^{tr} = F \text{ for all } g \in G\}.$$

Note that $(\mathbf{R}_{>0})\mathcal{F}_{>0}(G)$ is the set of G-invariant Euclidean scalar products on $\mathbf{R}^d$. G is called *uniform*, if there is essentially one G-invariant Euclidean structure on $\mathbf{R}^d$, that is if $\mathcal{F}_{>0}(G) = \{aF \mid 0 < a \in \mathbf{Q}\}$ for some $F \in M_d(\mathbf{Q})$.

(iii) $S = M_d(\mathbf{Q})$, and the action of H is conjugation: $S \times H \to S$, $(c,h) \mapsto h^{-1}ch$. Then the set of G-fixed points is the *commuting algebra* of G

$$C_{M_d(\mathbf{Q})}(G) := \{c \in M_d(\mathbf{Q}) \mid cg = gc \text{ for all } g \in G\}.$$

The following two remarks follow immediately from the normaliser principle.

REMARK 1. *Assume that G is uniform and let $F \in \mathcal{F}_{>0}(G)$. Then for each $n \in N$, the matrix nFn^{tr} is also G-invariant and therefore $nFn^{tr} = \left(\det(n)\right)^{2/d}F$. Hence n induces a similarity of F.*

REMARK 2. *For $n \in N$ and $L \in \mathcal{Z}(G)$, the lattice $Ln \in \mathcal{Z}(G)$ is also G-invariant.*

3. SIMILARITIES NORMALISE

In this section we show that if G is the automorphism group of a (strongly modular) lattice L then the similarities between L and $L' \in \pi(L)$ are elements of N.

PROPOSITION 3. *Let $G = \mathrm{Aut}(F) \leq GL_d(\mathbf{Z})$ be the full automorphism group of a lattice L. Assume that L is an integral lattice. Let $L' \in \pi(L)$ and $n \in GL_d(\mathbf{Q})$ which induces a similarity from L' to L, i.e. $L'n = L$ and $nFn^{tr} = aF$, ($a \in \mathbf{N}$). Then $a^{-1}n^2 \in G$ and $n \in N$.*

Proof. The matrix $a^{-1}n^2$ is clearly orthogonal with respect to F. Therefore to prove that $a^{-1}n^2 \in G$ we only have to show that $La^{-1}n^2 = L$. Now $L' = Ln^{-1}$, hence its dual lattice is

$$(L')^{\#} = \{v \in \mathbf{Q}^d \mid vF(ln^{-1})^{tr} \in \mathbf{Z} \quad \text{for all} \quad l \in L\}.$$

For $l \in L, v \in \mathbf{Q}^d$ we have $vF(ln^{-1})^{tr} = va^{-1}nFl^{tr}$ and hence $(L')^{\#} = L^{\#}an^{-1}$.

Since $L' \in \pi(L)$ one has $L' = L^{\#} \cap a^{-1}L$. Using this one obtains

$$Lan^{-2} = L'an^{-1} = L^{\#}an^{-1} \cap Ln^{-1} = (L')^{\#} \cap L' = L\,,$$

since $(L')^{\#}/L$ is the orthogonal complement of L'/L in $L^{\#}/L$ with respect to the induced quadratic form with values in $\mathbf{Q}/\mathbf{Z}$. So $a^{-1}n^2 \in G$.

Finally we check that $n \in N$. Let $g \in G$, then $n^{-1}gn$ is in $G = \mathrm{Aut}(F)$ since $Ln^{-1}gn = L'gn = L'n = L$ and

$$n^{-1}gnFn^{tr}g^{tr}n^{-tr} = n^{-1}agFg^{tr}n^{-tr} = F\,. \qquad \square$$

4. Obtaining Elements of N

Now we give examples as to how one may construct elements n of the normaliser N. To obtain similarities we are interested in $n \in N$ of determinant $\pm p^{d/2}$ for some (squarefree) natural number p such that $p^{-1}n^2 \in G$. The first method is an application of the normaliser principle to the situation (iii) described in Section 2:

PROPOSITION 4. *Let $U \trianglelefteq G$ be a normal subgroup of G and assume that the commuting algebra $K := C_{M_d(\mathbf{Q})}(U)$ is isomorphic to a number field. If $c \in K$ satisfies $c^2 = p \in \mathbf{Q}^*I_d$, then c lies in N.*

Proof. Since G normalises U, it acts by conjugation (and hence as Galois automorphisms) on the abelian number field K. Now let $c \in K$, with $c^2 =: p \in \mathbf{Q}^*I_d$ and $g \in G$. Then g stabilises the subfield $\mathbf{Q}[c]$ and hence $g^{-1}cg = \pm c$, which is equivalent to $c^{-1}gc = \pm g \in G$. Therefore $c \in N$, since we assumed that $-I_d \in G$. $\square$

The following construction described in [PlN 95] Proposition (II.4) also allows us to find elements of N.

For $i = 1, 2$ let $G_i \leq GL_{d_i}(\mathbf{Q})$ be finite rational irreducible matrix groups with commuting algebras $A_i \subseteq M_{d_i}(\mathbf{Q})$. Also let Q be a maximal common subalgebra of dimension z of A_1 and A_2. Let $d := \frac{d_1d_2}{z}$ and view the G_i as subgroups of $G_1 \otimes_Q G_2 \leq GL_d(\mathbf{Q})$. If there exist elements $a_i \in N_{GL_d(\mathbf{Q})}(G_i)$

centralising G_j and a_j $(1 \leq i \neq j \leq 2)$ and a squarefree natural number $p \neq 0$ such that $p^{-1}a_i^2 \in G_i$, the group

$$G := \langle G_1 \underset{Q}{\otimes} G_2, p^{-1}a_1a_2 \rangle ,$$

generated by the elements of $G_1 \underset{Q}{\otimes} G_2$ and $p^{-1}a_1a_2$, is a finite subgroup of $GL_d(\mathbf{Q})$ containing $G_1 \underset{Q}{\otimes} G_2$ as a subgroup of index 2.

For $d \leq 31$ and $p > 1$ we only need the case where a_2 is an element of the enveloping algebra of G_2. Then G is denoted by $G_1 \overset{2(p)}{\underset{Q}{\otimes}} G_2$ (or $G_1 \overset{2(p)}{\underset{Q}{\boxtimes}} G_2$) according to whether a_1 is (or is not) a rational linear combination of elements of G_1.

Using this notation one immediately has the following proposition.

PROPOSITION 5. *For $i = 1, 2$ the matrix a_i is an element of determinant $\pm p^{d/2}$ in the normaliser N of G.*

A common feature of the situations in Propositions 4 and 5 is that we extend the natural representation of G to a projective representation which is realisable as a linear representation over a quadratic extension of $\mathbf{Q}$.

PROPOSITION 6. *Let $G \trianglelefteq E$ be a supergroup containing G of index 2. Assume that $C_{M_d(\mathbf{Q})}(G) \cong \mathbf{Q}$ and that the natural character of G extends to E with character field $\mathbf{Q}[\sqrt{p}]$, where $p \in \mathbf{Z}$ is not a square. Then there exists $n \in N$ of determinant $\pm p^{d/2}$ with $p^{-1}n^2 \in G$.*

Proof. By Clifford theory one may extend the natural representation Δ of G to a representation $\delta_1 \otimes \delta_2 : E \to \left(\mathbf{Q}[\sqrt{p}] \otimes M_d(\mathbf{Q})\right)^*$, where δ_1 and δ_2 are projective representations $\delta_1(G) = \{1\}$ and $(\delta_2)_{|G} = \Delta$. Let $e \in E \setminus G$. Then

$$\left(\delta_1(e) \otimes \delta_2(e)\right)^2 = \delta_1(e)^2 \otimes \delta_2(e)^2 = 1 \otimes \Delta(e^2),$$

since $e^2 \in G$. Therefore $\delta_1(e)^2 \in \mathbf{Q}$. Replacing $\delta_1(e)$ by a suitable rational multiple (and multiplying $\delta_2(e)$ by the inverse) one may assume that $\delta_1(e)^2 = p^{-1}$. Then $n := \delta_2(e)$ is an element of the normaliser N with the desired properties. □

	Aut(L)	det(L)	min(L)	$\lvert L_{\min} \rvert$	lattice sparse
4	$F_4 \tilde{\otimes} F_4 = [2_+^{1+8}.O_8^+(2)]_{16}$	2^8	4	4320	+
6	$[(SL_2(9) \underset{\infty,3}{\overset{2(3)}{\otimes}} SL_2(9)).2]_{16}$	3^8	4	720	+
9	$[(Sp_4(3) \circ C_3) \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} SL_2(3)]_{16}$	$2^8 \cdot 3^8$	6	960	+
14	$[2.\mathrm{Alt}_{10}]_{16}$	5^8	6	2400	+
16	$[SL_2(5) \underset{\infty,2}{\overset{2(2)}{\boxtimes}} 2^{1+4'}.\mathrm{Alt}_5]_{16}$	$2^8 \cdot 5^8$	8	1200	+
19	$[SL_2(5) \underset{\infty,3}{\overset{2(3)}{\boxtimes}} SL_2(9)]_{16}$	$3^8 \cdot 5^8$	10	1440	+
21	$[SL_2(5) \underset{\infty,3}{\overset{2(3)}{\boxtimes}} (SL_2(3) \overset{2}{\square} C_3)]_{16}$	$2^8 \cdot 3^8 \cdot 5^8$	12	480	+
25	$[2.\mathrm{Alt}_7 \underset{\sqrt{-7}}{\overset{2(3)}{\boxtimes}} \tilde{S}_3]_{16}$	$3^8 \cdot 7^8$	12	1680	+
26	$[SL_2(7) \underset{\sqrt{-7}}{\overset{2(3)}{\boxtimes}} \tilde{S}_3]_{16}$	$3^8 \cdot 7^8$	10	336	$p \neq 2$
3	$[2.Co_1]_{24}$	1^{24}	4	196560	+
6	$[6.U_4(3).2 \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} SL_2(3)]_{24}$	2^{12}	4	3024	$p \neq 3$
16	$[6.L_3(4).2 \overset{2(2)}{\otimes} D_8]_{24}$	$2^{12} \cdot 3^{12}$	8	3024 + 7560	+
17	$[(SL_2(3) \circ C_4).2 \underset{\sqrt{-1}}{\overset{2(3)}{\boxtimes}} U_3(3)]_{24}$	$2^{12} \cdot 3^{12}$	8	4536 + 6048	+
18	A_{24}	1^{24}	2	600	$p \neq 5$
22	$[2.J_2 \overset{2}{\square} SL_2(5)]_{24}$	5^{12}	8	37800	+
35	$[L_2(7) \overset{2(2)}{\boxtimes} F_4]_{24}$	7^{12}	8	1008 + 3024	$p \neq 2$
40	$[SL_2(13) \overset{2(2)}{\square} SL_2(3)]_{24}$	13^{12}	12	$2 \cdot 2184 + 8736$	$p \neq 2$
42	$[6.\mathrm{Alt}_7 : 2]_{24}$	2^{12}	4	3024	+
43	$[3.M_{10} \underset{\sqrt{-3}}{\overset{2(2)}{\boxtimes}} SL_2(3)]_{24}$	$2^{12} \cdot 5^{12}$	8	1080	$p \neq 3$
44	$[\mathrm{Alt}_5 \underset{\sqrt{5}}{\overset{2}{\boxtimes}} (C_3 \overset{2(2)}{\boxtimes} D_8)]_{24}$	$2^{12} \cdot 3^{12} \cdot 5^{12}$	16	$360 + 2 \cdot 720$	$p \neq 2$
45	$[3.M_{10} \overset{2(2)}{\boxtimes} D_8]_{24}$	$2^{12} \cdot 3^{12} \cdot 5^{12}$	16	1080 + 1080	+
64	$[SL_2(11) \underset{\sqrt{-11}}{\overset{2(2)}{\boxtimes}} SL_2(3)]_{24}$	$2^{12} \cdot 11^{12}$	12	1320	$p \neq 2$

5. Proof of the Theorem

In this section we prove the theorem stated in the Introduction. The principle of the proof is given in the following remark.

REMARK 7. *Let* $G = \mathrm{Aut}(F) \leq GL_d(\mathbf{Z})$ *be a uniform automorphism group of the lattice* $L = \mathbf{Z}^d$ *and let* $n \in N$, *where* N *is the normaliser of* G. *Then by Remark 1,* n *induces a similarity* $L \to Ln$. *Remark 2 says that the lattice* Ln *is also* G*-invariant. Let* $L' \in \pi(L)$ *such that* $\det(n) = \left[L : (L \cap Ln)\right] \left[Ln : (L \cap Ln)\right]^{-1}$ *equals* $[L' : L]^{-1}$. *Then* $\left[L' : (L' \cap Ln)\right] = \left[Ln : (L' \cap Ln)\right]$. *So one may conclude that, if there is no other* G*-invariant lattice* M *in the layer of* L' *(i.e. with* $\left[L' : (M \cap L')\right] = \left[M : (M \cap L')\right]$*), then* $Ln = L'$.

The last uniqueness condition is fulfilled if $C_{M_d(\mathbf{Q})}(G) \cong \mathbf{Q}$, all lattices in $\mathcal{Z}(G)$ are even, and G is lattice sparse according to the following definition. In this case $\mathcal{Z}(G) = \{aL' \mid L' \in \pi(L),\ a \in \mathbf{Q}^*\}$ for any G-invariant lattice L. Note that this implies that the exponent $\exp(L^{\#}/L)$ is square free.

DEFINITION 8. *If* p *is a prime then a finite group* $G \leq GL_d(\mathbf{Q})$ *is called* p-lattice sparse *if any lattice* $L \in \mathcal{Z}(G)$ *can be obtained from any other lattice in* $\mathcal{Z}(G)$ *that contains* L *of* p*-power index by a combination of the five operations: taking sums, intersections, the dual lattice or the even sublattice with respect to some* $F \in \mathcal{F}_{>0}(G)$, *or multiplying by invertible elements of* $C_{M_d(\mathbf{Q})}(G)$. *The group* G *is called* lattice sparse *if* G *is* p*-lattice sparse for all primes* p.

Since the proof of the Theorem is similar for all r.i.m.f. groups, we only deal with the most interesting cases $d = 16$ and 24.

The r.i.m.f. groups of degree 16 and 24 fixing strongly modular lattices, which are not proper tensor products, are displayed in the table. The first column gives the number of the group under which it is referred to in [NeP 95] or [Neb 95] and [Neb 96]. The second column contains a name for the matrix group as partially explained in the paragraph preceding Proposition 5. In the notation there we additional make the following abbreviations. If $z = d_1$ or d_2, we omit $\times$ and Q in the symbols. Also (1) is omitted if $p = 1$. The division algebra Q is abbreviated as α, if $Q = \mathbf{Q}[\alpha]$, by the set of ramified primes, if Q is a quaternion algebra over $\mathbf{Q}$, and omitted if $Q = \mathbf{Q}$. For the finite simple and quasisimple groups we use the notation of [CCNPW 85] except that the alternating group is denoted by Alt_n to avoid confusion with A_n, which also denotes the automorphism group of the root lattice A_n.

The next three columns give information about the invariant strongly modular lattice L. First the determinant det(L) is given as the product of the abelian invariants of the Sylow subgroups of $L^\#/L$. The next column contains min(L) the minimum of the square lengths of the non zero vectors in L. From these two columns one may see whether L is an extremal lattice as defined in [Que 96]. The number of vectors of square length min(L) decomposed as a sum of the orbit lengths under Aut(L) is displayed in the fifth column. The last column contains the primes p for which Aut(L) is p-lattice sparse. A + indicates that Aut(L) is lattice sparse.

Proof of the Theorem. The commuting algebras of the groups in the table are all isomorphic to $\mathbf{Q}$ except for the one of $[6.\mathrm{Alt}_7 : 2]_{24}$ which is $\mathbf{Q}[\sqrt{-6}]$. So all these groups are uniform. Since the arguments are similar for all groups G we only deal with $G = \left[SL_2(5) \overset{2(3)}{\underset{\infty,3}{\boxtimes}} SL_2(9)\right]_{16}$ extensively. Let $L \in \mathcal{Z}(G)$ be a G-invariant lattice. There is a unique $F \in \mathcal{F}_{>0}(L)$ such that $\{l_1 F l_2^{tr} \mid l_1, l_2 \in L\} = \mathbf{Z}$. The determinant of L with respect to F is $|L^\#/L| = 3^8 \cdot 5^8$ and its minimum is 10. If this lattice is strongly modular, then it is an extremal strongly modular lattice ([Que 96]).

Since G is of the form described in Proposition 5 with $p = 3$, there is an element $a_1 \in N = N_{GL_{16}(\mathbf{Q})}(G)$ with $\frac{1}{3}a_1^2 \in G$. Since G is lattice sparse and $\det(a_1) = \frac{1}{3}^8$ one has $La_1 = 3L^\# \cap L \in \mathcal{Z}(G)$ and $a_1 F a_1^{tr} = 3F$ (by Remarks 2 and 1). Hence a_1 induces a similarity between L and $3L^\# \cap L$.

Next consider the normal subgroup $U := SL_2(5) \underset{\infty,3}{\otimes} SL_2(9) \trianglelefteq G$. The commuting algebra $C_{M_{16}(\mathbf{Q})}(U) =: K$ is isomorphic to $\mathbf{Q}[\sqrt{5}]$. From Proposition 4 one obtains an element $c \in N$ with $c^2 = 5$. As above one concludes that c yields a similarity between L and $5L^\# \cap L$. The product $a_1 c \in N$ is of determinant $\pm 15^8$ and gives a similarity between L and $15L^\#$. Therefore L is strongly modular.

Most of the other groups can be dealt with similarly. One has to use Proposition 6 to construct an additional element of N for the r.i.m.f. groups 4 and 14 of $GL_{16}(\mathbf{Q})$. For $G = [2.\mathrm{Alt}_{10}]_{16}$ (number 14), one obtains $n \in N$ of determinant $\pm 5^8$, since the character extends to $2.S_{10}$ with character field $\mathbf{Q}[\sqrt{\pm 5}]$ (cf. [CCNPW 85]). Analogously, for $F_4 \tilde{\otimes} F_4 = 2_+^{1+8}.O_8^+(2)$ (number 4) the character extends to $2_+^{1+8}.O_8^+(2).2$ with character field $\mathbf{Q}[\sqrt{\pm 2}]$.

The strong modularity for the lattices of the r.i.m.f. groups 9 and 21 of $GL_{16}(\mathbf{Q})$ (in particular the similarity of L with the lattice corresponding to the Sylow-2-subgroup of $L^\#/L$) may be derived from the equality

$$\left[(Sp_4(3) \circ C_3) \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} SL_2(3)\right]_{16} = \left[(Sp_4(3) \circ C_3) \underset{\sqrt{-3}}{\overset{2(2)}{\boxtimes}} SL_2(3)\right]_{16}$$

and

$$\left[SL_2(5) \underset{\infty,3}{\overset{2(3)}{\boxtimes}} (SL_2(3) \overset{2}{\square} C_3)\right]_{16} = \left[(SL_2(5).2 \circ C_3) \underset{\sqrt{-3}}{\overset{2(2)}{\boxtimes}} SL_2(3)\right]_{16}$$

using Proposition 5.

Similarly one uses Proposition 5 to show the 2-modularity of the lattices of the r.i.m.f. group 6 in $GL_{24}(\mathbf{Q})$ using the description

$$\left[6.U_4(3).2 \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} SL_2(3)\right]_{24} = \left[6.U_4(3).2 \overset{2(2)}{\circ} SL_2(3)\right]_{24}.$$

For the groups 44 and 64, which are the only groups which are not p-lattice sparse for a relevant prime p (=2), one has to note that the invariant sublattice of index 2^{12} in L is unique.

The theorem now follows from the next lemma. □

LEMMA 9. *The lattices (of determinant* $3^8 \cdot 5^8$*) of the r.i.m.f. subgroup* $G := [\pm \mathrm{Alt}_6 .2^2]_{16} \leq GL_{16}(\mathbf{Q})$ *(number 20 of* [NeP 95]*) are not (strongly) modular.*

Proof. Let L be such a G-invariant lattice and $L' \in \pi(L)$. Assume that there is a similarity $s : L' \to L$. By Proposition 3, this similarity s normalises G. Let $U \cong \mathrm{Alt}_6$ be the characteristic subgroup $\cong \mathrm{Alt}_6$ of G. Since the full automorphism group of U is already induced by conjugation with elements of G, there exists $g \in G$, such that $n := gs \in GL_{16}(\mathbf{Q})$ centralises U. Hence $n \in C_{M_{16}(\mathbf{Q})}(U) \cong \mathbf{Q}[\sqrt{5}]$. Since this number field does not contain an element of norm 3, one concludes that $[L' : L] = 5^8$. So the lattice L is neither similar to $L^\#$ nor to the lattice $L' \in \pi(L)$ corresponding to the 3-Sylow subgroup of $L^\#/L$. Note that if $[L' : L] = 5^8$, an element $x \in C_{M_{16}(\mathbf{Q})}(U)$ with $x^2 = 5$, induces a similarity by Proposition 4. □

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