1. Introduction

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THE NORMALISER ACTION AND STRONGLY MODULAR LATTICES

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ABSTRACT. We derive group theoretical methods to test if a lattice is strongly modular. We then apply these methods to the lattices of rational irreducible maximal finite groups.

1. INTRODUCTION

Let $L \subseteq \mathbf{R}^d$ be an even integral lattice in the Euclidean space of dimension d and let $L^{\#} \subseteq \mathbf{R}^d$ be its dual lattice. Let $\pi(L)$ be the set of all intermediate lattices $L \leq L' \leq L^{\#}$ that are inverse images of sums of Sylow subgroups of the finite abelian group $L^{\#}/L$. Then, L is said to be *strongly modular* if L is similar to L' for all $L' \in \pi(L)$ (cf. [Que 96]). Recall that L and L' are called *similar* if there exists $s \in GL(\mathbf{R}L)$ and $a \in \mathbf{R}_{>0}$ such that $Ls = L'$ and $(vs, ws) = a(v, w)$ for all $v, w \in \mathbf{R}L$, where $(\,,)$ denotes the Euclidean scalar product.

The automorphism group

$$G := \mathrm{Aut}(L) = \{g \in O(\mathbf{R}L) \mid Lg \subseteq L\}$$

is conjugate to a finite subgroup of $GL_d(\mathbf{Z})$. Since G acts as group automorphisms on $L^{\#}/L$ it preserves the lattices $L' \in \pi(L)$.

In Section 3 it is shown that the similarities $L' \to L$ normalise G. So one may use the normaliser

$$N_{GL_d(\mathbf{Q})}(G) := \{n \in GL_d(\mathbf{Q}) \mid n^{-1}gn \in G \quad \text{for all} \quad g \in G\}$$

*) Supported by the DFG.

of G in $GL_d(\mathbf{Q})$ to test strong modularity of L. In the next section we derive some methods for explicitly constructing elements of $N_{GL_d(\mathbf{Q})}(G)$.

Every finite subgroup of $GL_d(\mathbf{Q})$ is a subgroup of the automorphism group of an integral lattice. In particular the maximal finite subgroups of $GL_d(\mathbf{Q})$ are automorphism groups of distinguished lattices. A subgroup of $GL_d(\mathbf{Q})$ is called rational irreducible if it does not preserve a proper subspace $\neq \{0\}$ of $\mathbf{Q}^d$. The rational irreducible maximal finite, abbreviated to *r.i.m.f.*, subgroups of $GL_d(\mathbf{Q})$ are classified for $d < 32$ (cf. [PlN 95], [NeP 95], [Neb 95], [Neb 96], [Neb 96a]). Their invariant lattices provide many examples of strongly modular lattices. The following theorem is proved by applying the methods derived in Section 4.

THEOREM. *In dimension $d < 32$, all even lattices $L \subseteq \mathbf{R}^d$ that are preserved by a r.i.m.f. group and satisfy $L^{\#}/L \cong (\mathbf{Z}/l\mathbf{Z})^{d/2}$ for some $l \in \mathbf{N}$ are strongly modular, except for the lattices of the r.i.m.f. group $[\pm \mathrm{Alt}_6 .2^2]_{16}$ in $GL_{16}(\mathbf{Q})$ (cf.* [NeP 95]*).*

2. Preliminaries and Notation

The main strategy in this paper is the application of the following *normaliser principle.*

Let G be a group acting on a set S, H a subgroup of the group of transformations of S. Then the normaliser of G in H acts on the set of G-orbits.

In our situation $G = \mathrm{Aut}(L)$ is the automorphism group of an integral lattice L in the Euclidean space $\mathbf{R}L \cong \mathbf{R}^d$. By writing the action of G on $\mathbf{R}L$ with respect to a $\mathbf{Z}$-basis $(b_1, \dots, b_d)$ of L, G becomes a finite subgroup of $GL_d(\mathbf{Z})$. Then $G = \mathrm{Aut}(F) = \{g \in GL_d(\mathbf{Z}) \mid gFg^{tr} = F\}$ where F is the Gram matrix $F = \left((b_i, b_j)\right)_{i,j=1}^{d}$ of L.

For the rest of this article let $H = GL_d(\mathbf{Q})$, $G \leq H$, be a finite subgroup of H, and let $N := N_H(G)$ be its normaliser. We also assume that G contains the negative unit matrix, $-I_d \in G$.

We apply the normaliser principle to the following three situations.

(i) $S = \{L \subseteq \mathbf{Q}^d \mid L = \sum_{i=1}^{d} \mathbf{Z}b_i$ for a basis $(b_1, \dots, b_d)$ of $\mathbf{Q}^d\}$, the set of $\mathbf{Z}$-lattices of rank d in $\mathbf{Q}^d$, and the action of H on S is right multiplication: $S \times H \to S$, $(L, h) \mapsto Lh := \{lh \mid l \in L\}$. Then the set of G-fixed points is

$$\mathcal{Z}(G) := \{L \in S \mid Lg = L \text{ for all } g \in G\},$$

the set of G-invariant lattices.