# 1. Introduction

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# THE LOCAL LINEARIZATION PROBLEM FOR SMOOTH $SL(n)$-ACTIONS

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ABSTRACT. This paper considers $SL(n,\mathbf{R})$-actions on Euclidean space fixing the origin. We show that all $C^1$-actions on $\mathbf{R}^n$ are linearizable. We give $C^\infty$-actions of $SL(2,\mathbf{R})$ on $\mathbf{R}^3$ and of $SL(3,\mathbf{R})$ on $\mathbf{R}^8$ which are not linearizable. We classify the $C^0$-actions of $SL(n,\mathbf{R})$ on $\mathbf{R}^n$. Finally, the paper concludes with a study of the linearizability of $SL(n,\mathbf{Z})$-actions.

RÉSUMÉ. Dans cet article, on considère les actions de $SL(n,\mathbf{R})$ sur l'espace euclidien qui fixent l'origine. On montre que les actions $C^1$ sur $\mathbf{R}^n$ sont linéarisables. On donne des actions $C^\infty$ de $SL(2,\mathbf{R})$ sur $\mathbf{R}^3$ et de $SL(3,\mathbf{R})$ sur $\mathbf{R}^8$ qui ne sont pas linéarisables. On classifie les actions $C^0$ de $SL(n,\mathbf{R})$ sur $\mathbf{R}^n$. L'article s'achève par une étude de la linéarisabilité des actions de $SL(n,\mathbf{Z})$.

## 1. INTRODUCTION

If a group $G$ acts smoothly on a manifold $M$, fixing some point $x \in M$, then the differential of the action induces a linear action in the tangent space $T_xM$ to $M$ at $x$. The classical linearization problem is to determine whether the action of $G$ on $M$ is locally conjugate to its linear action on $T_xM$. In other words, is the action *linearizable* around $x$ ? In this paper we restrict ourselves largely to actions of $SL(n,\mathbf{R})$ on $\mathbf{R}^m$ fixing the origin: for brevity, we will simply say that $SL(n,\mathbf{R})$ acts on $(\mathbf{R}^m, 0)$.

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One of our results is:

THEOREM 1.1. *For all* $n > 1$ *and for all* $k = 1, \ldots, \infty$, *every* $C^k$*-action of* $SL(n, \mathbf{R})$ *on* $(\mathbf{R}^n, 0)$ *is* $C^k$*-linearizable.*

This result is not entirely unexpected. Indeed, D'Ambra and Gromov remarked that for actions of all semi-simple groups: "at least in the $C^{\text{an}}$-case (and probably in the $C^\infty$-case as well) the action is linearizable" [2, p. 98]. This was one of the main motivations of this present work. However, in [11], Guillemin and Sternberg gave an example of a $C^\infty$-action of the *Lie algebra* $\mathfrak{sl}(2, \mathbf{R})$ on $\mathbf{R}^3$ which is not linearizable (but which does not integrate to an action of $SL(2, \mathbf{R})$). They remarked: "(the linearization theorem) is false in the $C^\infty$ case unless some restrictions are placed on the algebra. What restrictions is unclear at present, but it seems that the algebra $\mathfrak{sl}(2, \mathbf{R})$ has to be singled out for special attention". Indeed, we show in Section 8 that Guillemin and Sternberg's example can be modified to give an action which integrates to a $C^\infty$-action of the *group* $SL(2, \mathbf{R})$ on $\mathbf{R}^3$ which is not linearizable (even topologically). However, the moral of our results is that linearizability is not so much a function of the algebra or the group, but of the dimension in which it acts. To further this claim, we give an example, in Section 9 below, of a $C^\infty$-action of $SL(3, \mathbf{R})$ on $\mathbf{R}^8$ which is also non-linearizable.

The paper is organized as follows. To put our results in context, we begin in Section 2 by recalling various classical linearization theorems. We state the linearization theorems of Bochner-Cartan, Sternberg and Kushnirenko, and we give proofs of the Bochner-Cartan theorem and Kushnirenko's theorem, since they are quite short. We recall Thurston's stability theorem, which we use repeatedly in this paper. We also give a proof of Hermann's result that smooth $SL(n, \mathbf{R})$-actions are formally linearizable.

In Section 3 we establish some preparatory results. In particular, we recall the notion of *suspension* (or *induction*). This is a procedure whereby, for a subgroup $H$ of a group $G$ and an action of $H$ on a space $M$, one extends the action to an action of $G$ on a bigger space $M' \supset M$ such that for each $x \in M$ the stabilizer of $x$ under the action of $G$ coincides with the stabilizer of $x$ under the original action of $H$. We then use this suspension procedure to prove two results which we require later in the paper, concerning $SO(n)$-actions.

The study of $SL(n, \mathbf{R})$ actions of $\mathbf{R}^n$ is done in two parts. The case $n \geq 3$ is treated in Section 4. We prove part of Theorem 1.1 here, and in the continuous case, *we give an explicit recipe for constructing all* $C^0$*-actions on* $\mathbf{R}^n$ : see Theorem 4.1.

In Section 5, we pause to recall some details of the adjoint representation of $SL(2,\mathbf{R})$ on its Lie algebra. Then in Section 6 we treat the linearizability of smooth $SL(2,\mathbf{R})$-actions on $\mathbf{R}^2$ and the classification of $C^0$-actions on $\mathbf{R}^2$. Actions of $SL(2,\mathbf{R})$ on $\mathbf{R}^m$ for $m > 2$ are quite prolific. We give *several examples* in Section 7. Then in Section 8 we give our *variation of Guillemin and Sternberg's example.* By using the method of suspension we show, in Section 9, that *one can construct a non-linearizable $C^\infty$-action of $SL(3,\mathbf{R})$ on $\mathbf{R}^8$*. This is also constructed from the adjoint representation.

The paper concludes in Section 10 with a study of the linearizability of $SL(n,\mathbf{Z})$-actions (and more generally of lattices in semi-simple groups). We show in particular:

THEOREM 1.2.

(a) *For no values of $n$ and $m$ with $n > m$, are there any faithful $C^1$-actions of $SL(n,\mathbf{Z})$ on $(\mathbf{R}^m, 0)$.*

(b) *There is a $C^\infty$-action of $SL(3,\mathbf{Z})$ on $(\mathbf{R}^8, 0)$ which is not topologically linearizable.*

(c) *There is a $C^\omega$-action of $SL(2,\mathbf{Z})$ on $(\mathbf{R}^2, 0)$ which is not linearizable.*

(d) *For all $n > 2$ and $m > 2$, every $C^\omega$-action of $SL(n,\mathbf{Z})$ on $(\mathbf{R}^m, 0)$ is $C^\omega$-linearizable.*

Throughout this paper, by a "$C^k$-action" we mean an action by $C^k$-diffeomorphisms which is continuous in the $C^k$-topology. To fix ideas, we make the following explicit definition:

DEFINITION 1.3. Consider a $C^1$-action $\Phi$ of a group $G$ on $(\mathbf{R}^m, 0)$ and simply denote by $g(x)$ the action of the element $g \in G$ on the point $x \in \mathbf{R}^m$. For $g \in G$, let $D(g) \in GL(m,\mathbf{R})$ denote the differential of the diffeomorphism $x \mapsto g(x)$ at the origin. Then $\Phi$ is *linearizable* if there are open neighbourhoods $U, V$ of the origin, and a homeomorphism $F\colon (U,0) \to (V,0)$, such that for each $g \in G$ the maps

$$x \mapsto F\big(g\big(F^{-1}(x)\big)\big) \quad \text{and} \quad x \mapsto D(g)\,(x)$$

have the same germ at the origin. If $\Phi$ and $F$ are $C^k$ (resp. $C^\infty$, resp. $C^\omega$) then we say that the action is $C^k$- (resp. $C^\infty$-, resp. $C^\omega$-) linearizable.

REMARK 1.4. Notice that "linearizable" really means "locally linearizable". We don't consider the question of global linearizability since, even under the strongest hypotheses, global linearizability is too much to expect. For example, the action by conjugation of $PSL(2,\mathbf{R})$ on its universal cover $\widetilde{SL}(2,\mathbf{R}) \cong \mathbf{R}^3$ is analytic and locally linearizable, by the exponential map of the Lie algebra, but it is not globally linearizable because it has countably many fixed points (corresponding to the infinite discrete centre). In fact, even for algebraic actions, global linearization is not guaranteed [38]. Throughout this paper we will use the word *local* to mean "in some neighbourhood of the origin". We make the point however that in the case of a locally linearizable action, each homeomorphism of the action has its own domain on which it is linearizable, but there may be no common open domain for the entire group.

Note that we could also deal with *local group actions*; that is, maps $\Phi$ from some open neighbourhood of $(\mathrm{Id}, 0) \in G \times \mathbf{R}^m$ to some neighbourhood of $0 \in \mathbf{R}^m$ which satisfy the same conditions as for actions but only in the neighbourhood of $(\mathrm{Id}, 0) \in G \times \mathbf{R}^m$. There would be no essential changes in what follows.

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## 2. BACKGROUND AND MOTIVATION

The introduction to [21] begins: "The subject of smooth transformation groups has been strongly influenced by the following two problems: the smooth linearization problem (Is every smooth action of a compact Lie group on Euclidean space conjugate to a linear action ?), and the smooth fixed point problem (Does every smooth action of a compact Lie group on Euclidean space have a fixed point ?)." Indeed, for *compact* group actions, one has the following theorem of Salomon Bochner and Henri Cartan:

BOCHNER-CARTAN THEOREM (see [30, Chap. V, Theorem 1]). *For all $k = 1, \ldots, \infty$, every $C^k$-action of a compact group $G$ on $(\mathbf{R}^m, 0)$ is $C^k$-linearizable.*

*Proof.* For each element $g \in G$, let $D(g)$ denote the differential of the action of $g$ at the origin. Consider the map $F\colon \mathbf{R}^m \to \mathbf{R}^m$, defined by