

## 4. $SL(n,R)$ -ACTIONS ON $\mathbb{R}^n$ FOR $n \geq 3$

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4.  $SL(n, \mathbf{R})$ -ACTIONS ON  $\mathbf{R}^n$  FOR  $n \geq 3$ 

Let  $n \geq 3$ . We first give examples of  $C^0$ -actions of  $SL(n, \mathbf{R})$  on  $\mathbf{R}^n$ . Consider the canonical projective action of  $SL(n, \mathbf{R})$  on  $S^{n-1}$ . Let  $\Delta^+$  be the radial half-line through the first basis element  $e_1$  and let  $H$  denote the subgroup of  $SL(n, \mathbf{R})$  that fixes  $\Delta^+$ . So  $SL(n, \mathbf{R})/H \cong S^{n-1}$ . Consider the homomorphism

$$\psi: (A_{ij}) \in H \mapsto \ln A_{11} \in \mathbf{R}.$$

Notice that one obtains a linear action of  $H$  on  $\mathbf{R}_*^+ = (0, \infty)$  by setting  $h(x) = e^{\psi(h)}x$ , for all  $h \in H$ ,  $x \in \mathbf{R}_*^+$ . Obviously this is conjugate to the  $H$ -action on  $\Delta^+$ . It follows from Lemma 3.7 that the action of  $SL(n, \mathbf{R})$  obtained by suspension of this action of  $H$  on  $\mathbf{R}_*^+$  is the canonical linear action of  $SL(n, \mathbf{R})$  on  $\mathbf{R}^n \setminus \{0\}$ . In fact, the map

$$\psi: [g \cdot x] \in (SL(n, \mathbf{R}) \times \mathbf{R}_*^+)/H \mapsto g(xe_1) \in \mathbf{R}^n \setminus \{0\}$$

is an isomorphism. We now deform the action of  $H$ . Choose a topological flow  $(\phi^t)_{t \in \mathbf{R}}$  on  $\mathbf{R}^+ = [0, \infty)$ , fixing 0. This defines an action of  $H$  on  $\mathbf{R}_*^+$  by setting  $h(x) = \phi^{\psi(h)}(x)$ , for all  $h \in H$ ,  $x \in \mathbf{R}_*^+$ . Now suspend this action of  $H$  and let  $\Phi$  denote the resulting action of  $SL(n, \mathbf{R})$  on the space  $M = (SL(n, \mathbf{R}) \times \mathbf{R}_*^+)/H$ . The space  $M$  fibres over  $S^{n-1}$ , with fibre  $\mathbf{R}_*^+$ , and the structure group is orientation preserving. So topologically,  $M$  is  $\mathbf{R}_*^+ \times S^{n-1}$ . Thus, identifying  $S^{n-1} \times \{0\}$  to a point, we obtain an  $SL(n, \mathbf{R})$ -action on  $\mathbf{R}^n$ . The fixed points of the flow  $\phi$  correspond to orbits in  $\mathbf{R}^n$  which are spheres of dimension  $n-1$ . In general, an  $n$ -dimensional orbit is either all of  $\mathbf{R}^n \setminus \{0\}$ , as in the linear case, or it is a spherical shell, bounded by  $S^{n-1}$  orbits, or a punctured ball bounded by an  $S^{n-1}$  orbit, or the exterior of an  $S^{n-1}$  orbit. In all cases, the  $n$ -dimensional orbits are conjugate to the canonical linear one on  $\mathbf{R}^n \setminus \{0\}$ , by Theorem 3.5(c).

**THEOREM 4.1.** *For all  $n \geq 3$ , every non-trivial  $C^0$ -action of  $SL(n, \mathbf{R})$  on  $(\mathbf{R}^n, 0)$  is conjugate to one of the above actions  $\Phi$ .*

*Proof.* Suppose that we have a non-trivial  $C^0$ -action of  $SL(n, \mathbf{R})$  on  $(\mathbf{R}^n, 0)$ . First use Proposition 3.8 to linearize the  $SO(n)$ -action. Then by Lemma 3.9, the  $SL(n, \mathbf{R})$ -action preserves the radial lines. Hence the radial projection  $\mathbf{R}^n \setminus \{0\} \rightarrow S^{n-1}$  is equivariant, where the action of  $SL(n, \mathbf{R})$  on  $S^{n-1}$  is the canonical projective one. Let  $H$  be the stabilizer of the radial half-line  $\Delta^+$  through  $e_1$ , as above. So the action of  $SL(n, \mathbf{R})$  on  $\mathbf{R}^n \setminus \{0\}$  is induced by some action of  $H$  on  $\mathbf{R}$ . Notice that this action is trivial when restricted to

$SO(n-1)$ . It remains to consider all actions of  $H$  on  $\mathbf{R}$  which are trivial on  $SO(n-1)$ . Again, by Lie [23, *ibid.*], these are given by homomorphisms from  $H$  to  $\mathbf{R}$ ,  $\mathbf{Aff}$ , or (some cover of)  $PSL(2, \mathbf{R})$ . We have the homomorphism  $\psi: (A_{ij}) \in H \mapsto \ln A_{11} \in \mathbf{R}$ . Note that  $\ker \psi = SL(n-1, \mathbf{R}) \ltimes \mathbf{R}^{n-1}$ . But it is easy to see that there are no non-trivial homomorphisms of  $\ker \psi$  to  $\mathbf{R}$  or  $\mathbf{Aff}$ . There are no non-trivial homomorphisms of  $\ker \psi$  to  $SL(2, \mathbf{R})$ , except in the case  $n=3$ , and in this case there are no such homomorphisms which are trivial on  $SO(n-1)$ . So the only possibility left is that  $H$  acts on  $\mathbf{R}$  by some flow. Finally, we put back the origin, as in the proof of Proposition 3.8. This completes the proof of the theorem.  $\square$

We now prove Theorem 1.1 for  $n \geq 3$ .

**THEOREM 4.2.** *For all  $n \geq 3$  and  $k = 1, \dots, \infty$ , every  $C^k$ -action of  $SL(n, \mathbf{R})$  on  $(\mathbf{R}^n, 0)$  is  $C^k$ -linearizable.*

*Proof.* Let  $n \geq 3$  and  $k = 1, \dots, \infty$  and suppose that we have a non-trivial  $C^k$ -action of  $SL(n, \mathbf{R})$  on  $(\mathbf{R}^n, 0)$ . By Remark 3.4, we may assume that the differential of the action at the origin is either the identity or the map  $g \mapsto (g^{-1})^t$ . We will assume that it is the identity; the other possibility can be handled using the same argument.

Linearizing the  $SO(n)$ -action, using the Bochner-Cartan theorem, one may assume that the  $SO(n)$ -action is the canonical one. Then by Lemma 3.9, the  $SL(n, \mathbf{R})$ -action preserves the radial lines. Let  $\Delta$  denote the radial line through the first of the canonical basis elements,  $e_1$ . Consider  $H = \text{Stab}_{SL(n, \mathbf{R})}(\Delta)$ , as before. So, as we saw in the proof of Theorem 4.1,  $H$  defines a  $C^k$ -flow on  $\Delta$ . This flow is hyperbolic, by the first paragraph. Hence by Theorem 2.5, this flow is linearizable by some local  $C^k$ -diffeomorphism  $f$  of  $\Delta(\cong \mathbf{R})$ . So, after conjugacy, we may assume that  $H$  acts linearly on  $\Delta$ . Now define the local  $C^k$ -diffeomorphism  $F$  of  $\mathbf{R}^n$  by the formula:

$$(2) \quad F(x) = \begin{cases} \frac{f(\|x\|)}{\|x\|} x, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

To see that  $F$  is of class  $C^k$ , the key point is to verify that  $f$  is a  $C^k$  odd function on  $\mathbf{R}$ . This follows easily from the fact that the flow on  $\Delta$  commutes with  $\text{Stab}_{SO(n, \mathbf{R})}(\Delta)$ , and the  $SO(n)$ -action is linear.

Now notice that  $F$  agrees with  $f$  on  $\Delta^+ = \{te_1 \in \Delta : t \geq 0\}$ , and as  $F$  commutes with the  $SO(n)$ -action, the  $SO(n)$ -action is unchanged by conjugation by  $F$ . In particular, the  $SO(n)$ -action still commutes with dilations.

It follows that after conjugation by  $F$ , the  $SL(n, \mathbf{R})$ -action commutes with dilations. Indeed, consider the conjugated  $SL(n, \mathbf{R})$ -action. If  $f \in SL(n, \mathbf{R})$ ,  $x \in \mathbf{R}^n$  and  $\lambda > 0$ , then choose  $a, b \in SO(n)$  such that  $ax \in \Delta^+$  and  $bf(\lambda x) \in \Delta^+$ . Provided  $x$  is sufficiently close to 0,  $ax$  and  $bf(\lambda x)$  will lie in the domain of  $f$ . Then  $bfa^{-1} \in H$  and so

$$\begin{aligned} f(\lambda x) &= b^{-1}bfa^{-1}a(\lambda x) = b^{-1}(bfg^{-1})\lambda a(x) \\ &= b^{-1}\lambda(bfa^{-1})a(x) = \lambda b^{-1}(bfa^{-1})a(x) \\ &= \lambda f(x). \end{aligned}$$

The proof of the theorem is then completed by the following well known result (cf. [17, Lemma 2.1.4]).  $\square$

LEMMA 4.3. *Every  $C^1$  map commuting with dilations is linear.*

*Proof.* Suppose that  $f$  is a  $C^1$ -diffeomorphism of  $\mathbf{R}^n$  which commutes with dilations. By comparing the differential of  $\lambda \cdot f$  and  $f \circ \lambda$  at  $x$  we have  $\lambda df|_x = \lambda df|_{\lambda x}$ , for each  $\lambda > 0$  and every  $x \in \mathbf{R}^n$ . Hence  $df|_x = df|_{\lambda x}$  and so  $df$  is constant on the radial lines. Thus  $df|_x = df|_0$  for all  $x$  and so  $f$  is linear.  $\square$

## 5. THE ADJOINT REPRESENTATION OF $SL(2, \mathbf{R})$

Let us recall some facts concerning the linear representations of  $SL(2, \mathbf{R})$ . Let  $P_l(\mathbf{R}^2)$  denote the space of real valued homogeneous polynomials, of two variables, of degree  $l$ . As a vector space,  $P_l(\mathbf{R}^2) \cong \mathbf{R}^{l+1}$ , and the action of  $SL(2, \mathbf{R})$  on  $\mathbf{R}^2$  defines a linear action on  $P_l(\mathbf{R}^2)$ : up to isomorphism, this is the (unique) irreducible representation of  $SL(2, \mathbf{R})$  in dimension  $l + 1$ . In dimension 3, there is another useful realization of the polynomial representation, called the *adjoint representation*. Notice that the group  $SL(2, \mathbf{R})$  acts by the adjoint representation on its Lie algebra  $\mathfrak{sl}(2, \mathbf{R})$ . Of course,  $\mathfrak{sl}(2, \mathbf{R})$  is the space of  $2 \times 2$  real traceless matrices; so as a vector space,  $\mathfrak{sl}(2, \mathbf{R}) \cong \mathbf{R}^3$ . The adjoint representation  $Ad: SL(2, \mathbf{R}) \rightarrow GL(3, \mathbf{R})$ , defined by

$$Ad(g): h \mapsto ghg^{-1}, \quad \forall g \in SL(2, \mathbf{R}), \quad h \in \mathfrak{sl}(2, \mathbf{R}),$$

is an irreducible linear representation. In fact, an explicit equivariant isomorphism  $\psi: \mathfrak{sl}(2, \mathbf{R}) \rightarrow P_2(\mathbf{R}^2)$  is obtained by taking  $\psi(h)$ , as a function of variables  $x$  and  $y$ , to be the area of the parallelogram spanned by  $(x, y)$  and  $h(x, y)$ . That is,