7. Examples of C^0-actions of SL(2,R) on R^m

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$$\Phi(A^t) = \Phi\left(\begin{pmatrix} 1 & b(e^{-2t}-1)/2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix}\right)$$
$$= \Phi\left(\begin{pmatrix} 1 & b(e^{-2t}-1)/2 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix}$$

since H acts linearly on the x-axis. Hence, since the family of maps

$$F_t = \Phi\left(\begin{pmatrix} 1 & b(e^{-2t}-1)/2 \\ 0 & 1 \end{pmatrix}\right) \qquad t \geq 0$$

is equicontinuous in some neighbourhood of the identity, we conclude that Σ_A is the x-axis, as required.

By the above argument, we may assume that locally the $SO(2)$-action is the canonical one and the $SL(2,\mathbf{R})$-action preserves the radial lines. The proof is then completed as in the proof of Theorem 4.2. □

7. Examples of C^0-actions of $SL(2,\mathbf{R})$ on $\mathbf{R}^m$

When m is greater than n there is a plethora of examples of continuous actions of $SL(n,\mathbf{R})$ on $(\mathbf{R}^m, 0)$. In this section we give some examples in the case $n = 2$.

7.1. The symmetric product. Choose one of the continuous $SL(2,\mathbf{R})$-actions on $(\mathbf{R}^2, 0)$ from the previous section. Now consider the associated $SL(2,\mathbf{R})$-action on the symmetric product

$$\Pi_{i=1}^m \mathbf{R}^2/\Sigma_m \cong \mathbf{C}^m,$$

where Σ_m is the symmetric group on m letters. Recall that the last identification associates to an m-tuple of points $(x_1, \dots, x_m)$ in $\mathbf{R}^2 \cong \mathbf{C}$ the coefficients of the monic polynomial of degree m in one complex variable whose roots are the x_i. As the original action fixed the origin in $\mathbf{R}^2$, so the corresponding action fixes the origin in $\mathbf{R}^{2m}$.

7.2. The adjoint action at infinity. Consider the adjoint action of $SL(2,\mathbf{R})$ on $\mathbf{R}^3$, as discussed in Section 5. Removing the origin and compactifying the other end, we obtain a C^0-action of $SL(2,\mathbf{R})$ on $\mathbf{R}^3$, which we will call the *adjoint action at infinity.* This action is certainly not topologically linearizable, since all the orbits now accumulate to the fixed point. In fact, this action is not topologically conjugate to any C^1-action. To see this, consider the hyperbolic element $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Using the exponential

$\exp(th)$, one obtains a one-parameter subgroup in $SL(2,\mathbf{R})$ which, by the adjoint action, defines a flow $\mathfrak{F}$ on $\mathfrak{sl}(2,\mathbf{R})$. Choose the following basis for $\mathfrak{sl}(2,\mathbf{R}) \cong \mathbf{R}^3$:

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}.$$

Then a simple computation shows that the flow $\mathfrak{F}$ is generated by the vector field $X = z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}$ (where (x,y,z) are the coordinates with respect to the above basis). Restricted to each plane $x = constant$, the vector field X has a standard hyperbolic singularity, with index -1, and on the invariant lines $z = -y$ and $z = y$, the flow is contracting and expanding respectively. It follows that if the $SL(2,\mathbf{R})$-action at infinity was C^1, then the differential at infinity of the action of X would be trivial. In this case, the differential at infinity of the entire $SL(2,\mathbf{R})$-action would be trivial, contradicting Thurston's stability theorem.

7.3. THE ACTION ON THE CLOSED SUBGROUPS OF $\mathbf{R}^2$. Recall that from [35] the space Gr of closed subgroups of $\mathbf{R}^2$, with the Hausdorff topology, is homeomorphic to S^4. Obviously $SL(2,\mathbf{R})$ acts continuously on Gr, and the two trivial subgroups, $\{0\}$ and $\mathbf{R}^2$, are fixed by this action. Inside Gr there is an invariant S^3 comprised of the set K of subgroups isomorphic to $\mathbf{R}$, together with the set of subgroups isomorphic to $\mathbf{Z}^2$ which have generators which span a parallelogram of area 1. The set K, which is a trefoil knot in S^3, is a 1-dimensional orbit, and its complement $S^3 - K$ is a single 3-dimensional orbit.

Removing one of the fixed subgroups, $\{0\}$ or $\mathbf{R}^2$, one obtains an interesting $SL(2,\mathbf{R})$-action on $\mathbf{R}^4$ with one fixed point. Notice that this action is not conjugate to a C^1-action. Indeed, if the action was C^1, then the differential at the origin would define a linear representation of $SL(2,\mathbf{R})$ in $\mathbf{R}^4$. So this representation would be a direct sum of irreducible representations. Since $-\mathrm{Id}$ acts trivially on Gr, it follows that it is either the sum of the canonical 3-dimensional representation with the trivial 1-dimensional representation, or it is the trivial 4-dimensional representation. But the second case is not possible, by Thurston's stability theorem. In the first case, one could linearize the $SO(2)$-action, using the Bochner-Cartan theorem, and thus locally one would find a 2-dimensional subspace through the origin which was fixed pointwise by $SO(2)$. But there are no closed subgroups of $\mathbf{R}^2$ which are $SO(2)$-invariant, apart from $\{0\}$ and $\mathbf{R}^2$. So this case is also impossible.

7.4. CONING ACTIONS ON SPHERES. If one has a non-trivial $SL(2,\mathbf{R})$-action on S^m, then taking the cone in the obvious sense, one obtains an $SL(2,\mathbf{R})$-action on $(\mathbf{R}^{m+1},0)$. We claim that such actions cannot be conjugate to C^1 actions. Indeed, actions defined by coning have invariant spheres around 0. If a C^1 diffeomorphism has a family of invariant topological spheres around the origin, it cannot have any stable manifold so that all the eigenvalues of its differential at the origin have modulus one. No non-trivial linear representation of $SL(2,\mathbf{R})$ has the property that all eigenvalues of all elements have modulus one. So, if the action under consideration was C^1 the differential at the origin would be trivial: this is a contradiction with Thurston's stability theorem.

There are many interesting actions of $SL(2,\mathbf{R})$ on spheres. Compactifying the actions of Section 6 gives examples on S^2. An action on S^3 was given in Example 7.3. Notice also that if one has actions of $SL(2,\mathbf{R})$ on S^p and S^q, then there is an associated action of $SL(2,\mathbf{R})$ on their join $S^p * S^q = S^{p+q+1}$.

Finally we remark that many interesting actions of $SL(n,\mathbf{R})$ on spheres, for $n \geq 3$, can be found in the papers of Fuichi Uchida (see for example [46, 47, 48]).

8. A C^∞-ACTION OF $SL(2,\mathbf{R})$ WHICH IS NOT LINEARIZABLE

Here we give a variation of the Guillemin-Sternberg example a C^∞-action of the Lie algebra $\mathfrak{sl}(2,\mathbf{R})$ on $\mathbf{R}^3$ which is not linearizable. The action we give below integrates to a C^∞ non-linearizable $SL(2,\mathbf{R})$-action. It is obtained by deforming the adjoint action of $SL(2,\mathbf{R})$ on its Lie algebra. The constructed action is clearly non-linearizable since it has an orbit of dimension 3.

By differentiation, the adjoint action of $SL(2,\mathbf{R})$ defines a Lie algebra $\mathfrak{g}$ (isomorphic to $\mathfrak{sl}(2,\mathbf{R})$) of vector fields on $\mathbf{R}^3$. This algebra can be explicitly computed as follows: choose an element $h \in \mathfrak{sl}(2,\mathbf{R})$, take its exponential $\exp h$, and compute the derivative of the adjoint map $Ad\big(\exp(t\,h)\big)$ at $t=0$. A convenient basis for $\mathfrak{g}$ is:

$$X = z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}, \quad Y = z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}, \quad R = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$$

Here R is the derivative of $Ad\big(\exp(t\,h)\big)$ where $h = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The commutator relations are:

$$[X,Y] = -R, \quad [R,X] = Y, \quad [R,Y] = -X.$$