8. A C^∞ -action of SL(2,R) which is not linearizable

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7.4. CONING ACTIONS ON SPHERES. If one has a non-trivial $SL(2,\mathbf{R})$-action on S^m, then taking the cone in the obvious sense, one obtains an $SL(2,\mathbf{R})$-action on $(\mathbf{R}^{m+1},0)$. We claim that such actions cannot be conjugate to C^1 actions. Indeed, actions defined by coning have invariant spheres around 0. If a C^1 diffeomorphism has a family of invariant topological spheres around the origin, it cannot have any stable manifold so that all the eigenvalues of its differential at the origin have modulus one. No non-trivial linear representation of $SL(2,\mathbf{R})$ has the property that all eigenvalues of all elements have modulus one. So, if the action under consideration was C^1 the differential at the origin would be trivial: this is a contradiction with Thurston's stability theorem.

There are many interesting actions of $SL(2,\mathbf{R})$ on spheres. Compactifying the actions of Section 6 gives examples on S^2. An action on S^3 was given in Example 7.3. Notice also that if one has actions of $SL(2,\mathbf{R})$ on S^p and S^q, then there is an associated action of $SL(2,\mathbf{R})$ on their join $S^p * S^q = S^{p+q+1}$.

Finally we remark that many interesting actions of $SL(n,\mathbf{R})$ on spheres, for $n \geq 3$, can be found in the papers of Fuichi Uchida (see for example [46, 47, 48]).

8. A C^∞-ACTION OF $SL(2,\mathbf{R})$ WHICH IS NOT LINEARIZABLE

Here we give a variation of the Guillemin-Sternberg example a C^∞-action of the Lie algebra $\mathfrak{sl}(2,\mathbf{R})$ on $\mathbf{R}^3$ which is not linearizable. The action we give below integrates to a C^∞ non-linearizable $SL(2,\mathbf{R})$-action. It is obtained by deforming the adjoint action of $SL(2,\mathbf{R})$ on its Lie algebra. The constructed action is clearly non-linearizable since it has an orbit of dimension 3.

By differentiation, the adjoint action of $SL(2,\mathbf{R})$ defines a Lie algebra $\mathfrak{g}$ (isomorphic to $\mathfrak{sl}(2,\mathbf{R})$) of vector fields on $\mathbf{R}^3$. This algebra can be explicitly computed as follows: choose an element $h \in \mathfrak{sl}(2,\mathbf{R})$, take its exponential $\exp h$, and compute the derivative of the adjoint map $Ad\big(\exp(t\,h)\big)$ at $t=0$. A convenient basis for $\mathfrak{g}$ is:

$$X = z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}, \quad Y = z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}, \quad R = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$$

Here R is the derivative of $Ad\big(\exp(t\,h)\big)$ where $h = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The commutator relations are:

$$[X,Y] = -R, \quad [R,X] = Y, \quad [R,Y] = -X.$$

The idea is now to deform this action by adding in a component in the direction of the radial vector field:

$$\mathbf{r} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}.$$

We don't change R, but we set $\bar{X} = X + f\mathbf{r}, \bar{Y} = Y + g\mathbf{r}$, for some functions f and g and we want to impose the same relations as before:

$$[\bar{X}, \bar{Y}] = -R, \quad [R, \bar{X}] = \bar{Y}, \quad [R, \bar{Y}] = -\bar{X}.$$

Since $\mathbf{r}$ commutes with X, Y and R, this requires

(3) $$R(f) = g$$

(4) $$R(g) = -f$$

(5) $$X(g) - Y(f) + f\mathfrak{r}(g) - g\mathfrak{r}(f) = 0.$$

Equations (3) and (4) give $R^2 f + f = 0$, which suggests that one looks for functions of the form

$$f(x, y, z) = xA(z, \sqrt{x^2 + y^2}),$$

for some function $A\colon \mathbf{R}^2 \to \mathbf{R}$. Then $g(x, y, z) = -yA(z, \sqrt{x^2 + y^2})$ and equation (5) gives

$$X(yA) + Y(xA) = 0.$$

This has the smooth solution

$$A(z, v) = \frac{a(v^2 - z^2)}{v^2},$$

where $a\colon \mathbf{R} \to \mathbf{R}$ is any C^∞-function which is zero on $\mathbf{R}^-$. It follows that for each choice of a, the vector fields

$$\bar{X} = X + xA(z, \sqrt{x^2 + y^2})\mathfrak{r}, \quad \bar{Y} = Y - yA(z, \sqrt{x^2 + y^2})\mathfrak{r}, \quad \text{and} \quad R$$

generate a Lie algebra $\mathfrak{A}$ isomorphic to $\mathfrak{sl}(2, \mathbf{R})$. By choosing a to be a bounded function, we guarantee that the elements of $\mathfrak{A}$ are complete vector fields. Indeed, take a Riemannian metric on $\mathbf{R}^3$ which, outside the unit ball, is $g_{(x,y,z)}/\sqrt{x^2 + y^2 + z^2}$, where $g_{(x,y,z)}$ denotes the standard Euclidean metric. This defines a complete Riemannian metric, with respect to which the elements of $\mathfrak{A}$ are bounded. Hence, by [1, Proposition 2.1.21] for example, the elements of $\mathfrak{A}$ are complete. By integration, we consequently obtain a smooth action of the universal cover of $SL(2, \mathbf{R})$ whose orbits are those of $\mathfrak{A}$. In fact, since we haven't changed the definition of R, this gives a smooth action of $SL(2, \mathbf{R})$ whose orbits are those of $\mathfrak{A}$.

Notice that the vector fields $\overline{X}, \overline{Y}, R$ are linearly independent wherever $a \neq 0$. Indeed, putting $v = \sqrt{x^2 + y^2}$, one has:

$$\det(\overline{X}, \overline{Y}, R) = \det \begin{pmatrix} x^2A(z,v) & z + xyA(z,v) & y + xzA(z,v) \\ z - xyA(z,v) & -y^2A(z,v) & x - yzA(z,v) \\ -y & x & 0 \end{pmatrix}$$

$$= -(v^2 - z^2)\, v^2 A(z,v) = -(v^2 - z^2)\, a(v^2 - z^2)\,.$$

It follows that if the function a is non-zero on $\mathbf{R}^+$, then the set of hyperbolic points in $\mathfrak{sl}(2,\mathbf{R})$ constitute a single orbit under the new action of $SL(2,\mathbf{R})$. Since no linear action of $SL(2,\mathbf{R})$ in $\mathbf{R}^3$ has an orbit of dimension 3, we conclude that our new action of $SL(2,\mathbf{R})$ is not linearizable. Note that outside the open orbit, this action coincides with the adjoint linear action.

In order to motivate the construction that we shall present in the next section, we now present another way of describing the non-linearizable action that we just constructed. Consider the subgroup Diag of $SL(2,\mathbf{R})$ of diagonal matrices and consider the trivial action of Diag on the positive line $\mathbf{R}^+_*$. It is easy to see that the suspension of this action is conjugate to the adjoint action of $SL(2,\mathbf{R})$ outside the invariant cone in $\mathbf{R}^3$. Now, since Diag is isomorphic to $\mathbf{R} \times \mathbf{Z}/2\mathbf{Z}$, it is easy to let Diag act non-trivially on $\mathbf{R}^+_*$ and the new suspension will provide a new action of $SL(2,\mathbf{R})$. If the new action of Diag extends to $\mathbf{R}^+$ and is sufficiently flat at 0, this action of $SL(2,\mathbf{R})$ can be equivariantly glued to the invariant cone and provides non-linearizable smooth actions of $SL(2,\mathbf{R})$ on $(\mathbf{R}^3, 0)$.

9. A C^∞-ACTION OF $SL(3,\mathbf{R})$ WHICH IS NOT LINEARIZABLE

We start with the adjoint action of $SL(3,\mathbf{R})$ on its Lie algebra $\mathfrak{sl}(3,\mathbf{R}) \cong \mathbf{R}^8$. Denote by Diag the subgroup of $SL(3,\mathbf{R})$ of diagonal matrices. This group is isomorphic to $\mathbf{R}^2 \times (\mathbf{Z}/2\mathbf{Z})^2$. Let *diag* $\subset \mathfrak{sl}(3,\mathbf{R})$ denote the 2-dimensional subalgebra consisting of diagonal matrices. The Weyl group, which is in this case the symmetric group on 3 letters, acts linearly on *diag* by permutation of the axis. The orbit of any point in *diag* under the adjoint action is a properly embedded submanifold of $\mathfrak{sl}(3,\mathbf{R})$ which intersects *diag* on some orbit of the Weyl group. Let C be a Weyl chamber in *diag*, for example the region consisting of diagonal matrices $(\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_1 \leq \lambda_2 \leq \lambda_3$. This is a fundamental domain for the action of the Weyl group.