

# 4. POLYGONS WITH GIVEN SIDES – KÄHLER STRUCTURES

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section collapsed. The trace function on  $\mathcal{M}_{m \times 2}(\mathbf{C})$  descends to  $\tilde{\mathbf{G}}_2(\mathbf{C}^m)$  and to the Casimir function “perimeter” on  ${}^m\mathcal{P}\mathcal{P}_+^3$ .

4. POLYGONS WITH GIVEN SIDES – KÄHLER STRUCTURES

We now use the map  $\ell : {}^m\tilde{\mathcal{P}}^k, {}^m\mathcal{P}_+^k, {}^m\mathcal{P}^k \rightarrow \mathbf{R}^m$  defined in (2.4). Recall that  $\ell(\rho)$ , for  $\rho \in {}^m\tilde{\mathcal{P}}^k$ , is the length of the successive sides of a representative of  $r$  with total perimeter 2.

For  $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbf{R}_{\geq 0}^m$  with  $\sum_{i=1}^m \alpha_i = 2$ , we define

$${}^m\tilde{\mathcal{P}}^k(\alpha) :=: \tilde{\mathcal{P}}^k(\alpha) := \{\rho \in {}^m\tilde{\mathcal{P}}^k \mid \ell(\rho) = \alpha\} \subset {}^m\tilde{\mathcal{P}}^k.$$

The space  $\tilde{\mathcal{P}}^k(\alpha)$  is invariant under the action of  $O_k$ . We define the moduli spaces

$$\mathcal{P}_+^k(\alpha) := SO_k \backslash \tilde{\mathcal{P}}^k(\alpha) = \ell^{-1}(\alpha) \subset {}^m\mathcal{P}_+^k$$

and

$$\mathcal{P}^k(\alpha) := O_k \backslash \tilde{\mathcal{P}}^k(\alpha) = \ell^{-1}(\alpha) \subset {}^m\mathcal{P}^k.$$

The space  $\tilde{\mathcal{P}}^1(\alpha)$  consists of a finite number of points and is generically empty. We call  $\alpha$  *generic* if  $\tilde{\mathcal{P}}^1(\alpha) = \emptyset$ .

**THEOREM 4.1.** *The map  $\mu := \ell \circ \widehat{\Phi} : \mathbf{G}_2(\mathbf{C}^m) \rightarrow \mathbf{R}^m$  is a moment map for the action of  $U_1^m$  on  $\mathbf{G}_2(\mathbf{C}^m)$ .*

*Proof.* As seen in (3.13), the moment map  $\Psi : \mathbf{G}_2(\mathbf{C}^m) \rightarrow \mathcal{H}(m)$  for the  $U_m$ -action on  $\mathbf{G}_2(\mathbf{C}^m)$  is induced from  $\tilde{\Psi} : \mathcal{M}_{m \times 2}(\mathbf{C}) \rightarrow \mathcal{H}(m)$  given by  $\tilde{\Psi}(a, b) := (a, b) \cdot (a, b)^*$ . A moment map  $\mu$  for the action of  $U_1^m$  is obtained by composing  $\Psi$  with the projection  $\mathcal{H}(m) \rightarrow \mathbf{R}^m$  associating to a matrix its diagonal entries. So, if  $\Pi \in \mathbf{G}_2(\mathbf{C}^m)$  is generated by  $a$  and  $b$  with  $(a, b) \in \mathbf{V}_2(\mathbf{C}^m)$ , one has

$$\mu(\Pi) = (|a_1|^2 + |b_1|^2, \dots, |a_m|^2 + |b_m|^2) = \ell \circ \widehat{\Phi}(a, b). \quad \square$$

A now classic theorem of Atiyah and Guillemin-Sternberg [Au, §III.4.2] asserts that the image of a moment map for a torus action is a convex polytope (the *moment polytope*). The restriction of the moment map to the fixed point set of an anti-symplectic involution has the same image [Du]. In our case, one gets these facts directly :

COROLLARY 4.2. *The moment map  $\mu: \mathbf{G}_2(\mathbf{C}^m) \longrightarrow \mathbf{R}^m$  satisfies  $\mu(\mathbf{G}_2(\mathbf{C}^m)) = \mu(\mathbf{G}_2(\mathbf{R}^m)) = \Xi_m$ , where  $\Xi_m$  is the hypersimplex*

$$\Xi_m := \{(x_1, \dots, x_m) \in \mathbf{R}^m \mid 0 \leq x_i \leq 1 \text{ and } \sum_{i=1}^m x_i = 2\}.$$

*Proof.* One has  $\text{Image}(\mu) = \text{Image}(\ell)$ . Further it is manifest that  $\text{Image}(\ell) \subset \Xi_m$ . A proof that  $\text{Image}(\ell) = \Xi_m$  is actually provided in [KM1], Lemma 1, or [Ha]. We give here however another argument, for the pleasure of constructing a continuous section  $\sigma: \Xi_m \longrightarrow {}^m\mathcal{P}^2$  of  $\ell$ . If  $m = 3$ , we have already mentioned in (2.7) that  ${}^3\mathcal{P}^2$  is homeomorphic to  $\Xi_3$  via the map  $\ell$ . Let  $\alpha \in \Xi_m$ . Define  $\beta_i := \sum_{j=1}^i \alpha_j$  and

$$r(\alpha) := \min\{i \mid \beta_i \leq 1 \text{ and } \beta_{i+1} \geq 1\}.$$

The numbers  $\beta_r, \alpha_r, 2 - \beta_{r+1}$  form a triple of  $\Xi_3$  and are then the lengths of a unique triangle  $\tau(\alpha) \in {}^3\mathcal{P}^2$ , which can be subdivided in the obvious way to define the element  $\sigma(\alpha) \in {}^m\mathcal{P}^2(\alpha)$  (see Figure 1).

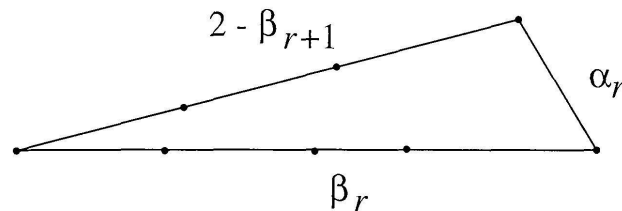


FIGURE 1:  $\tau(\alpha)$

The continuity of  $\sigma$  comes from the fact that if the map  $r$  is discontinuous at some  $\alpha$ , the triangle  $\tau(\alpha)$  is then lined.  $\square$

REMARKS. 1) Corollary 4.2 is also a consequence of our stronger result (5.4).

2) The word “hypersimplex” is introduced in [GM]. Observe that  $H$  is obtained by taking the convex hull of the middle point of each edge of a standard  $(m - 1)$ -simplex.

We also obtain the critical values of  $\mu$  (compare [Ha]):

PROPOSITION 4.3. *The set of critical values of  $\mu$  on  $\mathbf{G}_2(\mathbf{C}^m) \rightarrow \Xi_m$  or  $\mathbf{G}_2(\mathbf{R}^m) \rightarrow \Xi_m$  consists of those points  $(x_1, \dots, x_m) \in \Xi_m$  satisfying one of the following conditions:*

- a) *one  $x_i$  vanishes;*
- b) *one  $x_i$  is equal to 1;*
- c) *there exist  $\varepsilon_i = \pm 1$  such that  $\sum_{i=1}^m \varepsilon_i x_i = 0$ , with at least two  $\varepsilon_i$ 's of each sign.*

REMARK. Points satisfying a) and b) constitute the boundary of  $\Xi_m$ . Points satisfying c) are “inner walls”. Points satisfying a) correspond to non-proper polygons. Those satisfying b) or c) are non-generic  $\alpha$ 's (Condition b) implies that there exist  $\varepsilon_i = \pm 1$  such that  $\sum_{i=1}^m \varepsilon_i x_i = 0$  with all but one  $\varepsilon_i$  of the same sign.)

*Proof.* The critical points of the moment map  $\mu$  are the points of  $\mathbf{G}_2(\mathbf{C}^m)$  for which the  $U_1^m$ -action has a stabilizer of dimension bigger than 1. They are the images of those  $(2 \times m)$ -matrices in  $\mathbf{V}_2(\mathbf{C}^m)$  for which the  $(U_1^m \times_{U_1} U_2)$ -action has a non-discrete stabilizer. There are such points whose stabilizer is contained in  $U_1^m \times \{1\}$ ; they are the matrix with one row vanishing and their values under  $\mu$  are the points of  $\Xi_m$  satisfying a). The other points give rise to points in  ${}^m\tilde{\mathcal{P}}^3 = U_1^m/\mathbf{V}_2(\mathbf{C}^m)$  so that the action of  $U_2/\{\text{center of } U_2\} \simeq SO_3$  has non discrete stabilizer. Those points are the lined configurations  ${}^m\tilde{\mathcal{P}}^1$ . Their values in  $\Xi_m$  are the non generic  $\alpha$ 's, which are the points in  $\Xi_m$  satisfying b) or c).  $\square$

We have proven most of the main result of this section: for generic and proper  $\alpha$ , the space  $\mathcal{P}^3(\alpha)$  is a Kähler sub-quotient of  $\mathbf{G}_2(\mathbf{C}^m)$ .

THEOREM 4.4. *For  $\alpha \in \text{int } \Xi_m$  generic,  $\mathcal{P}_+^3(\alpha)$  is a Kähler manifold isomorphic to the Kähler reduction  $U_1^m \setminus \mu^{-1}(\alpha)$ . The involution  $\smile$  is antiholomorphic and  $\mathcal{P}^2(\alpha)$  can be seen as the real part of  $\mathcal{P}_+^3(\alpha)$ .*

*Proof.* By 4.1, one has  $\mathcal{P}^3(\alpha) = \ell^{-1}(\alpha) = U_1^m \setminus \mu^{-1}(\alpha)$  and we have seen in 3.9 that  $\widehat{\Phi}(\bar{a}, \bar{b}) = \Phi(a, b)^\smile$ .  $\square$

We shall now compare the Kähler structure obtained on  $\mathcal{P}_+^3(\alpha)$  from the Grassmannian to that introduced by Klyachko [Kl] or Kapovich-Millson ([KM2], §3). Using the standard cross product  $\times$  and scalar product  $\langle \cdot, \cdot \rangle$  on  $\mathbf{R}^3$ , these authors put on the sphere  $S_r^2$  of radius  $r$  the complex structure  $\tilde{J}$

defined by

$$\tilde{J}v := \frac{1}{r} x \times v \quad (v \in T_x S_r^2)$$

and the Kähler metric

$$\tilde{h}(u, v) := \frac{1}{r} \langle u, v \rangle - \frac{i}{r^2} \langle x, u \times v \rangle \quad (u, v \in T_x S_r^2)$$

with associated symplectic form  $\tilde{\omega}(u, v) := \langle \frac{x}{r^2} u \times v \rangle$ . Let  $W(\alpha) := \prod_{i=1}^m S_{\alpha_i}^2$ . The map  $\beta : W_\alpha \rightarrow \mathbf{R}^3$  defined by  $\beta(z_1, \dots, z_m) := \sum_{i=1}^m z_i$  is the moment map for the diagonal action of  $SO_3$  on  $W_\alpha$ . The space  $\mathcal{P}_+^3(\alpha)$  thus occurs as the symplectic reduction  $SO_3 \backslash \beta^{-1}(0)$ .

PROPOSITION 4.5. *The complex structure  $J$  and Kähler metric  $h$  of 4.4 compare with those  $\tilde{J}$  and  $\tilde{h}$  of Kapovich-Millson in the following way:*

$$\tilde{J} = J \quad \text{and} \quad \tilde{h}(u, v) = 4h(u, v).$$

*Proof.* Starting from the Hermitian vector space  $\mathcal{M} = \mathcal{M}_{m \times 2}(\mathbf{C})$  one sees that  $\mathcal{P}^3(\alpha)$  is obtained by two successive symplectic reductions

$$\mathbf{G}_2(\mathbf{C}^m) = \tilde{\Phi}^{-1}(0)/U_2 \quad \text{and} \quad \mathcal{P}^3(\alpha) = U_1^m \backslash \mu^{-1}(\alpha)$$

(we use the notation of §3). One can perform the reductions in the reverse order. We first get

$$U_1^m \backslash \tilde{\Psi}^{-1}(\alpha) = \prod_{i=1}^m \mathbf{C}P_{\alpha_i}^1$$

where  $\mathbf{C}P_r^1$  is the quotient of the 3-dimensional sphere

$$\{(u, v) \in \mathbf{C}^2 \mid |u|^2 + |v|^2 = r\}$$

by the diagonal action of  $U_1$ . The moment map  $\tilde{\Phi} : \mathcal{M} \rightarrow \mathcal{H}(2)$  gives a moment map (still called  $\tilde{\Phi}$ ) from the product of projective spaces into  $\mathcal{H}_0(2)$ . One has a commutative diagram

$$\begin{array}{ccc} \prod_{i=1}^m \mathbf{C}P_{\alpha_i}^1 & \xrightarrow[\simeq]{\Pi^\phi} & \prod_{i=1}^m S_{\alpha_i}^2 \\ \tilde{\Phi} \downarrow & & \downarrow \beta \\ \mathcal{H}_0(2) & \xrightarrow[\simeq]{\psi} & \mathbf{R}^3 \end{array}$$

where  $\psi : \mathcal{H}_0(2) \rightarrow \mathbf{R}^3 \simeq \mathbf{R} \times \mathbf{C}$  sends the matrix  $\begin{pmatrix} u & z \\ \bar{z} & -u \end{pmatrix}$  to  $(u, z)$ .

To prove Proposition 4.5, it is enough to establish that for all  $a \in \mathbf{C}P_r^1$ , the tangent map  $T_a\phi : T_a\mathbf{C}P_r^1 \longrightarrow T_{\phi(a)}S_r^2$  satisfies

$$T_a\phi(Jv) = \tilde{J}T_a\phi(v) \quad \text{and} \quad \tilde{\omega}(T_a\phi(v), T_a\phi(Jv)) = 4\omega(v, Jv).$$

By  $U_2$ -equivariance, we can restrict ourselves to  $a = [\sqrt{r}, 0]$ . The tangent space  $T_a\mathbf{C}P_r^1$  is identified with  $\{0\} \times \mathbf{C}$  and one can take  $v = (0, 1)$  and  $Jv = (0, i)$ . One has  $\phi(a) = (r, 0, 0)$ ,

$$T_a\phi(v) = (0, 2\sqrt{r}, 0), \quad T_a\phi(Jv) = (0, 0, 2\sqrt{r}) = \tilde{J}T_a\phi(v)$$

and  $\tilde{\omega}(T_a\phi(v), T_a\phi(Jv)) = 4$ , while  $\omega(v, Jv) = 1$ .  $\square$

REMARKS

(4.6) The results of this section show that the spaces  $\mathcal{P}_+(\alpha)$  for generic  $\alpha$  are the symplectic leaves of the Poisson structure on the regular part of  ${}^m\mathcal{P}_+^3$ , or  ${}^m\mathcal{P}\mathcal{P}_+^3$  given in (3.13) and (3.14).

(4.7) If one works in the pure quaternions  $I\mathbf{H}$ , the complex structure  $\tilde{J}$  on  $S_r^2$  becomes

$$\tilde{J}(v) = \frac{qv}{|q|}, \quad (v \in T_qS_r^2 = I\mathbf{H}).$$

The sphere  $S_r^2$  is a co-adjoint orbit of  $U_1(\mathbf{H})$  and the Hermitian form  $\tilde{\omega}$  is the Kirillov–Kostant form (see [Gu, Theorem 1.1]).

(4.8) The isomorphism between the symplectic reductions of the Grassmannian  $\mathbf{G}_2(\mathbf{C}^m)$  and the product of  $\mathbf{C}P^1$ 's that underlies our results 3.9, 4.4 and the proof of 4.5 is a symplectic version of the Gel'fand-MacPherson correspondence ([GM] and [GGMS]). The fact that this isomorphism comes from two reductions of  $\mathcal{M}$  is the philosophy of “dual pairs” (see [Mo] and the references therein).

5. THE GEL'FAND-CETLIN ACTION

On  ${}^m\mathcal{F}^k$  we have so far defined the length functions  $\tilde{\ell}$  measuring the distances between successive vertices. We now introduce  $\tilde{d} : {}^m\mathcal{F}^k \rightarrow \mathbf{R}^m$ ,  $\tilde{d}(\rho) = (|\rho(1)|, |\rho(1) + \rho(2)|, \dots, |\sum_{i=1}^m \rho(i)|)$ , the lengths of the diagonals connecting the vertices to the origin. (Only  $m - 3$  of these functions are new, as  $\tilde{d}(\rho)_1 = \tilde{\ell}(\rho)_1$ ,  $\tilde{d}(\rho)_{m-1} = \tilde{\ell}(\rho)_m$ , and  $\tilde{d}(\rho)_m = 0$ . Hereafter we write only  $\ell_i, d_i$  and the  $\rho$  is to be understood.)