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THE REPRESENTATION THEORY OF AFFINE TEMPERLEY-LIEB ALGEBRAS

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ABSTRACT. We define a sequence $\mathbf{T}^a(n)$ $(n = 0, 1, 2, 3, \dots)$ of infinite dimensional algebras as the sets of endomorphisms of the objects in a certain category of diagrams. These algebras are extended versions of the Temperley-Lieb quotients of the affine Hecke algebras of type $\widetilde{A}_{n-1}$. They have bases consisting of diagrams drawn without intersections on the surface of a cylinder. Using the methods of cellular algebras, we construct certain finite dimensional representations of these algebras, which we call "cell" or "Weyl" modules; these come from "functors on the category of diagrams" and are therefore constructed simultaneously for all $\mathbf{T}^a(n)$.

There are canonical invariant bilinear forms which put pairs of the cell modules in duality with each other and all the irreducible $\mathbf{T}^a(n)$-modules are obtained as quotients of the cell modules by the radicals of the forms. By determining all homomorphisms between the cell modules, we are able to determine their decomposition matrices and from these to deduce the dimensions of all the irreducibles.

We also give explicit formulae for the discriminants of the forms. The representations we construct may be interpreted as representations of the affine Hecke algebra of type A; they therefore give explicit results about some of the representations of the affine Hecke algebra at roots of unity. Our results may also be applied to study related finite dimensional algebras such as the usual Temperley-Lieb algebra or Jones' annular algebra. For these, our results concerning discriminants give precise criteria for semisimplicity as well as a complete discussion of their modular representation theory, including the determination of the composition factors, with their multiplicities, of the cell modules. As a by-product of our explicit determination of the homomorphisms between the cell modules, we also obtain a closed formula for the Jones (or augmentation) idempotent of the Temperley-Lieb algebra which yields a presentation of Jones' projection algebra when the Jones trace on the Temperley-Lieb algebra is degenerate.

§0. Introduction and preliminaries

Let R be a commutative ring and let $q \in R$ be an invertible element. For any integer $n \geq 1$ let $H_n(q)$ be the Hecke algebra of type A_{n-1}, with standard generators $T_1, \dots, T_{n-1}$ and let $H_n^a(q) \supset H_n(q)$ be the corresponding affine Hecke algebra. Thus $H_n^a(q)$ has generators $T_1, \dots, T_n$ which satisfy

$$\text{(0.1a)} \qquad \begin{aligned} T_iT_j &= T_jT_i && \text{if } |i-j| \geq 2 \text{ and } \{i,j\} \neq \{1,n\}, \\ T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1} && i = 1, \dots, n, \text{ the subscripts being taken mod } n. \end{aligned}$$

and

$$\text{(0.1b)} \qquad (T_i - q)(T_i + q^{-1}) = 0.$$

It is well known that $H_n^a(q)$ has an R-basis consisting of elements T_w, where w runs over the elements of the affine Weyl group W^a of type $\widetilde{A}_{n-1}$ and that $H_n(q)$ is the free R-submodule with basis $\{T_w \mid w \in W\}$, where W is the Weyl group of type A_{n-1}. The T_w are defined as follows. Let $r_1, \dots, r_n$ be the set of simple reflections for W^a which correspond to the T_i. If $\ell(w)$ is the corresponding length function on W^a, for any $w \in W^a$ take a reduced expression $w = r_{i_1} \dots r_{i_\ell}$ ($\ell = \ell(w)$); then $T_w = T_{i_1} \dots T_{i_\ell}$.

For each $i = 1, \dots, n$ the group $W(i)$ generated by r_i and r_{i+1} is isomorphic to the symmetric group $Sym(3)$ (where i is taken modulo n). Thus we may form the quasi-idempotents

$$\text{(0.2)} \qquad E_i = \sum_{w \in W(i)} q^{\ell(w)} T_w$$

for $i = 1, \dots, n$. These satisfy

$$\text{(0.3)} \qquad E_i^2 = (\sum_{w \in W(i)} q^{2\ell(w)}) E_i = (1 - q^2)(1 - q^4) E_i.$$

Let $I_n^a = (E_1, \dots, E_n)$ be the ideal of $H_n^a(q)$ which is generated by $E_1, \dots, E_n$. It is well known (cf. e.g. [J1]) that the Temperley-Lieb algebra $\mathbf{T}(n) = TL_n(\delta)$ (where $\delta = -(q + q^{-1})$) may be defined by

$$\text{(0.4)} \qquad TL_n = H_n(q)/I_n$$

where $I_n = I_n^a \cap H_n(q)$ is the ideal of $H_n(q)$ generated by $E_1, \dots, E_{n-1}$.

Its affine analogue is $TL_n^a(\delta)$, defined by

$$\text{(0.5)} \qquad TL_n^a(\delta) = H_n^a(q)/I_n^a.$$

In this work we shall study algebras $\mathbf{T}^a(n)$ which are slightly larger than TL_n^a (see (2.9) below). They are obtained from TL_n^a by adding a "twist" (denoted τ below). It is these algebras (the $\mathbf{T}^a(n)$) which we refer to as the *affine Temperley-Lieb algebras*. We shall define the algebras in terms of what we shall call the *Temperley-Lieb category* $\mathbf{T}^a$ (see §2 below), whose objects are the non-negative integers $\mathbf{Z}_{\geq 0}$ and whose morphisms $\mathbf{T}^a(n, m)$ $(n, m \in \mathbf{Z}_{\geq 0})$ are R-linear combinations of "affine diagrams" from n to m (see (1.3) below). The algebra $\mathbf{T}^a(n)$ is then just $\mathbf{T}^a(n, n)$. We shall show (in (2.9) below) that the algebras defined in this way contain the Temperley-Lieb quotients TL_n^a of the affine Hecke algebra (cf. [Ch] or [Lu2]).

If W is any functor from $\mathbf{T}^a$ to the category R-**mod** of R-modules, then $W(n)$ is a $\mathbf{T}^a(n)$-module (for $n \in \mathbf{Z}_{\geq 0}$). We shall construct such functors $W_{t,z}$ for each pair (t, z) such that $t \in \mathbf{Z}_{\geq 0}$ and z is an invertible element of R. For each such pair (t, z) we shall define an invariant bilinear form

$$\langle\ .\ \rangle_{t,z} \colon W_{t,z} \times W_{t,z^{-1}} \to R$$

which is "generically" non-degenerate. When R is a field, we show that if $\mathrm{rad}_{t,z}$ is the radical of $\langle\ .\ \rangle_{t,z}$, then $L_{t,z} := W_{t,z}/\mathrm{rad}_{t,z}$ is absolutely irreducible and all irreducible finite dimensional $\mathbf{T}^a(n)$-modules are of this form.

We then give a characterisation of all homomorphisms between the cell modules (Theorem (3.4) below). This is in some sense the main result of this paper, as it enables us to determine the decomposition matrices of the cell modules. Section 4 is concerned with the determination of the discriminants of the forms $\langle\ .\ \rangle_{t,z}$. This gives explicit results concerning the semisimplicity of related finite dimensional algebras. In §5 we give the composition multiplicities of the components of the cell modules and derive corresponding statements for the related finite dimensional algebras. In (3.7) we derive, as a by-product of our explicit determination of the homomorphisms between the cell modules, a closed formula for the Jones or augmentation idempotent (see [MV], [We] and [Li]) of the Temperley-Lieb algebra when q is a root of unity. In [MV] certain coefficients of this idempotent are computed, while in [We] a recursive formula is given for it. Our formula (see (3.7) below) differs from these by being explicit and closed, although it only applies when q is a root of unity. It leads to a presentation of Jones' projection algebra when the Jones trace on the Temperley-Lieb algebra is not non-degenerate. This includes those values of q for which the Temperley-Lieb algebra is not semisimple. Since our algebras contain quotients of the affine Hecke algebras of type A, their representations yield representations of the affine Hecke algebras. Hence our results may be interpreted as a contribution to the representation theory of these affine Hecke algebras at roots of unity (see [KL1, KL2]). Our construction of representations of the algebras through functors on the category of diagrams may share some ideas with [FY] or [RT], although we have no heuristic explanation for the fact that all irreducible representations arise functorially (Theorem (2.8) below).

TOTALLY ORDERED SETS. In order to define the diagrams below, we introduce some constructions associated with totally ordered sets.

(0.6) If X and Y are totally ordered sets, form a new totally ordered set $X\#Y$ as follows. The underlying set is the disjoint union $X \amalg Y$ of X and Y. Let $\ell: X \to X \amalg Y$ and $u: Y \to X \amalg Y$ denote the canonical injections. Define the total order by stipulating that (in increasing order) the elements of (the image) $\ell(X)$ come first in reverse order, followed by the elements of $u(Y)$ in natural order. Intuitively, $X\#Y$ should be imagined as two horizontal lines in the real affine plane with the elements of X (identified with $\ell(X)$) on the lower line and those of Y (identified with $u(Y)$) on the upper line. The ordering is given by moving leftwards along the bottom line, then right along the top. An element of $X\#Y$ is said to be *lower* (or a lower vertex) if it lies in $\ell(X)$ and *upper* (or an upper vertex) if it lies in $u(Y)$.

(0.7) Given any totally ordered set X, form a new totally ordered set with a distinguished automorphism as follows. Denote by $\mathbf{Z} \times X$ the set of pairs (i,x) where i is an integer and $x \in X$ and order this set lexicographically: $(i_1,x_1) < (i_2,x_2)$ if $i_1 < i_2$ or $i_1 = i_2$ and $x_1 < x_2$. Define the automorphism V_X of $\mathbf{Z} \times X$ by $(i,x) \mapsto (i+1,x)$ $(i \in \mathbf{Z}, x \in X)$. If Y is another totally ordered set, then we may extend the permutations V_X and V_Y to the automorphism $V_X\#V_Y$ of $(\mathbf{Z} \times X)\#(\mathbf{Z} \times Y)$, given by $\ell(i,x) \mapsto \ell(i+1,x)$ for i in $\mathbf{Z}$ and x in X and $u(i,y) \mapsto u(i+1,y)$ for i in $\mathbf{Z}$ and y in Y. When there is no danger of confusion, we shall abbreviate V_X and $V_X\#V_Y$ to V.

We shall require the following result.

(0.8) LEMMA (cf. [GL, (4.5)]). *Let V be a permutation of a set X and assume that X has finitely many V orbits. Let ϕ_1 and ϕ_2 be involutory permutations of X which commute with V. Assume that the fixed point sets* $\operatorname{fix}(\phi_1)$ *and* $\operatorname{fix}(\phi_2)$ *are disjoint. Then the orbits $\mathcal{O}$ of the subgroup H (of permutations of X) generated by ϕ_1 and ϕ_2, fall into the following mutually exclusive classes:*

(1) $\mathcal{O}$ *contains no points in* $\mathrm{fix}(\phi_1) \cup \mathrm{fix}(\phi_2)$ *; in this case we call* $\mathcal{O}$ *a* loop.

(2) $\mathcal{O}$ *contains exactly two points in* $\mathrm{fix}(\phi_1) \cup \mathrm{fix}(\phi_2)$ *; in this case we call* $\mathcal{O}$ *an* arc *and refer to the two fixed points as the* ends *of the arc.*

When the ends of an arc (case (2) above) are not in the same set $\mathrm{fix}(\phi_i)$ ($i = 1$ or 2) we say the orbit is a *through arc.*

Proof. Suppose that an orbit $\mathcal{O}$ contains a point x of $\mathrm{fix}(\phi_1)$ (say). Following [GL, (4.5)], write $(\phi_1\phi_2)_i = \dots\phi_2\phi_1\phi_2$ (i factors) and write $x_i = (\phi_1\phi_2)_i x$ (for $i = 0, 1, 2, \dots$), so that $x_0 = x$ etc. Then clearly $\mathcal{O} = \{x_0, x_1, \dots\}$. If the orbit $\mathcal{O}$ is finite, the result is immediate by the argument in [GL, loc. cit.]. If $\mathcal{O}$ is infinite, then two of its elements lie in the same V-orbit by finiteness, whence there are indices $i < j$ and $k \in \mathbf{Z}$ such that $V^k x_i = x_j$. Acting by ϕ_1 and ϕ_2, it follows that $V^k x_0 = x_{j \pm i} = x_{r'}$ for some $r > 0$. Hence $x_{r'}$ is fixed by ϕ_1. It follows, using the same argument as in [GL, loc. cit.] that $\mathcal{O} = \{x_0, \dots, x_r\}$, which contradicts the infinite nature of $\mathcal{O}$. $\square$

Notice that the proof of (0.8) shows that any infinite H-orbits must be loops. Also, if X is finite, V may (and generally will) be trivial.

§ 1. Involutions, diagrams and categories

We shall consider various categories in this work whose objects are the non-negative integers $\mathbf{Z}_{\geq 0}$. The morphisms in these categories are defined in terms of "diagrams" and their "composition", whose definition in turn depends on the notion of a "planar involution" (cf. [GL, § 6]). In this section we develop a calculus of involutions and diagrams; our principal purpose is the definition of the category $\mathbf{D}^a$ of affine diagrams. These generalise the familiar diagrams which may be used to define the ordinary Temperley-Lieb algebra $\mathbf{T}(n)$.

(1.1) Definition.

(1) A *planar involution* of the totally ordered set P is a permutation ϕ of P such that ϕ^2 is the identity, ϕ has no fixed points and if $x, y \in P$ then $x \leq y \leq \phi(x) \Rightarrow x \leq \phi(y) \leq \phi(x)$.

(2) If t and n are non-negative integers, a *finite diagram* $\alpha\colon t \to n$ is a planar involution ϕ_α of $\mathbf{t\#n}$, where the latter set is defined in (0.6).

If we visualize $\mathbf{t\#n}$ as two horizontal lines in the plane as indicated in (0.6), such a diagram may be represented by a graph with vertex set $\mathbf{t\#n}$ and edges $(x, \phi(x))$ $(x \in \mathbf{t\#n})$. The planar condition then ensures that this graph can be drawn without intersections in the convex hull of $\mathbf{t\#n}$.

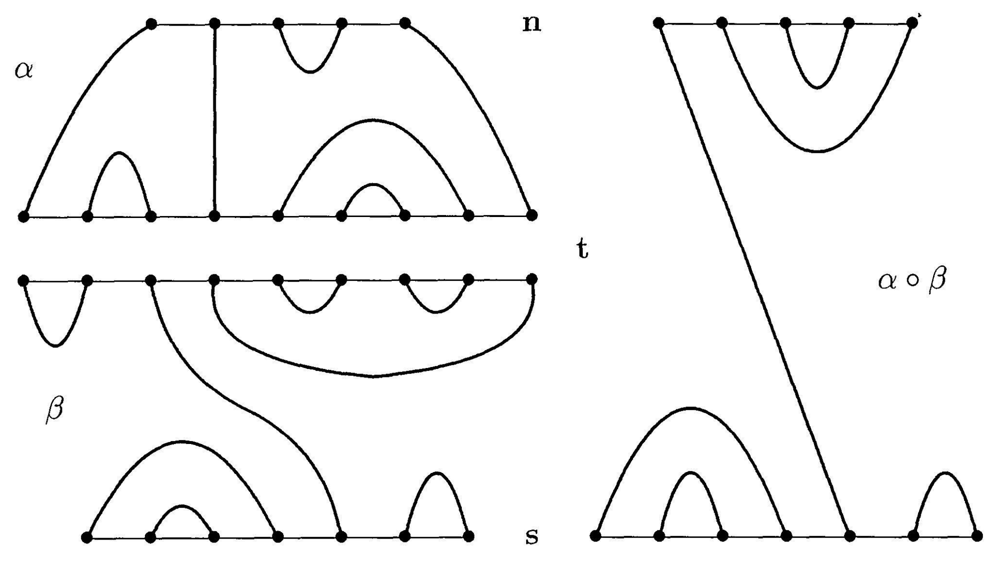
Suppose $\alpha : t \to n$ and $\beta : s \to t$ are two finite diagrams with corresponding planar involutions ϕ_α and ϕ_β of $\mathbf{t\#n}$ and $\mathbf{s\#t}$ respectively. We identify $\mathbf{s} \amalg \mathbf{n} = \ell(\mathbf{s}) \cup u(\mathbf{n})$ with its image in $\mathbf{s} \amalg \mathbf{t} \amalg \mathbf{n}$ using the canonical injection. Let $\widetilde{\phi}_\alpha$ denote the involutory bijection of $X = \mathbf{s} \amalg \mathbf{t} \amalg \mathbf{n}$ which fixes $\ell(\mathbf{s})$ and agrees with ϕ_α in the sense that $\widetilde{\phi}_\alpha \circ i_{23} = i_{23} \circ \phi_\alpha$ where $i_{23} : \mathbf{t} \amalg \mathbf{n} \to \mathbf{s} \amalg \mathbf{t} \amalg \mathbf{n}$ denotes the canonical injection. Similarly we obtain the involution $\widetilde{\phi}_\beta$ whose fixed point set is $u(\mathbf{n})$. This sets up the situation of (0.8) with $V = \mathrm{id}$.

(1.2) Definition.

(1) With the above notation, let $\phi_{\alpha\circ\beta}$ be the involution of $\mathbf{s\#n}$ which interchanges the ends of the arcs (0.8) of $H = \langle \widetilde{\phi}_\alpha, \widetilde{\phi}_\beta \rangle$ on $\mathbf{s} \amalg \mathbf{t} \amalg \mathbf{n}$. This is a planar involution. Define the *composition* $\alpha \circ \beta$ of α and β to be the diagram corresponding to this involution.

(2) Maintaining the notation of (1), denote by $m(\alpha, \beta)$ the number of loops of H on $\mathbf{s} \amalg \mathbf{t} \amalg \mathbf{n}$. Then $m(\alpha, \beta) = x - (s + n)/2$ where x is the total number of orbits.

In terms of the graphical representation of the diagrams, composition corresponds to placing a graph for α above a graph for β, identifying corresponding points indexed by vertices in $\mathbf{t}$ and deleting the $m(\alpha, \beta)$ interior loops formed. We give an example below.



We define the *category* **D** *of finite diagrams* as follows. Its objects are the non-negative integers. If $t, n \in \mathbf{Z}_{\geq 0}$, the morphisms from t to n are the finite diagrams $\alpha: t \to n$ and composition is as defined in (1.2). The *identity* $\mathrm{id}: t \to t$ interchanges $u(i)$ and $\ell(i)$ for $i \in \mathbf{t}$, in the notation of (0.6).

Next we extend the concept of diagram to the affine case. Let n be a non-negative integer. Recall from (0.7) that $\mathbf{Z} \times \mathbf{n}$ is ordered lexicographically and has an automorphism V_n. The orbits of V_n are represented by the elements of the subset $\{0\} \times \mathbf{n}$.

(1.3) DEFINITION. Let t and n be non-negative integers. An *affine diagram* $\alpha: t \to n$ is a pair $(g(\alpha), \phi_\alpha)$ where $g(\alpha)$ is a non-negative integer and ϕ_α is a planar involution of $(\mathbf{Z} \times \mathbf{t})\#(\mathbf{Z} \times \mathbf{n})$ which commutes with the shift $V_t\#V_n$ (see (0.7)) and which is such that when $g(\alpha)$ is nonzero, ϕ_α preserves the subsets $\ell(\mathbf{Z} \times \mathbf{t})$ and $u(\mathbf{Z} \times \mathbf{n})$.

An affine diagram $\alpha: t \to n$ may be thought of as a graph drawn without intersections on the surface of a cylinder. The lower and upper boundaries of the cylinder have vertices which are labelled by $\ell(\{0\} \times \mathbf{t})$ and $u(\{0\} \times \mathbf{n})$ respectively. Each vertex is joined to another one, the joining curve wrapping around the cylinder a certain number of times, this number being determined by ϕ_α; $g(\alpha)$ denotes the number of closed curves which wrap around the cylinder. The condition that there be no intersections means that if there are any such curves, no top vertex is joined to a bottom vertex (cf. the definition above). In practice it is more convenient to lift such graphs to the universal covering strip of the cylinder, which is the origin of the definition (1.3). We now explain this in detail.

Draw a rectangle (the "fundamental rectangle") in the real plane and extend the horizontal sides indefinitely. Label t points on the lower boundary (avoiding corners) of the rectangle in the obvious way by $\ell(\{0\} \times \mathbf{t})$ and n points on the upper boundary by $u(\{0\} \times \mathbf{n})$. Then label the translates of these points in the translates of the fundamental rectangle to the right and left by $\ell(\mathbf{Z} \times \mathbf{t})$ and $u(\mathbf{Z} \times \mathbf{n})$ in the obvious way. The resulting strip provides a graphical model for $(\mathbf{Z} \times \mathbf{t})\#(\mathbf{Z} \times \mathbf{n})$ (see (0.7)). It is covered by translates of the fundamental rectangle and the shift $V = V_t\#V_n$ moves the fundamental rectangle one step to the right. An affine diagram is depicted in this context by an "augmented graph", drawn without intersections in the strip. This consists of curves joining distinct vertices which are interchanged by ϕ_α, as well as $g(\alpha)$ horizontal curves which stretch along the whole strip, the latter representing closed curves on the cylinder. This graph must be fixed by the translation V. In practice, we draw only the part of the graph in the fundamental rectangle, which determines it completely due to the invariance under V. For the sake of simplicity, the lower and upper vertices of the fundamental rectangle will be labelled in our figures by $\mathbf{t}$ and $\mathbf{n}$ respectively, rather than by the more formal $\ell(\{0\} \times \mathbf{t})$ and $u(\{0\} \times \mathbf{n})$. We shall also use this notation in the text when there is no danger of confusion.

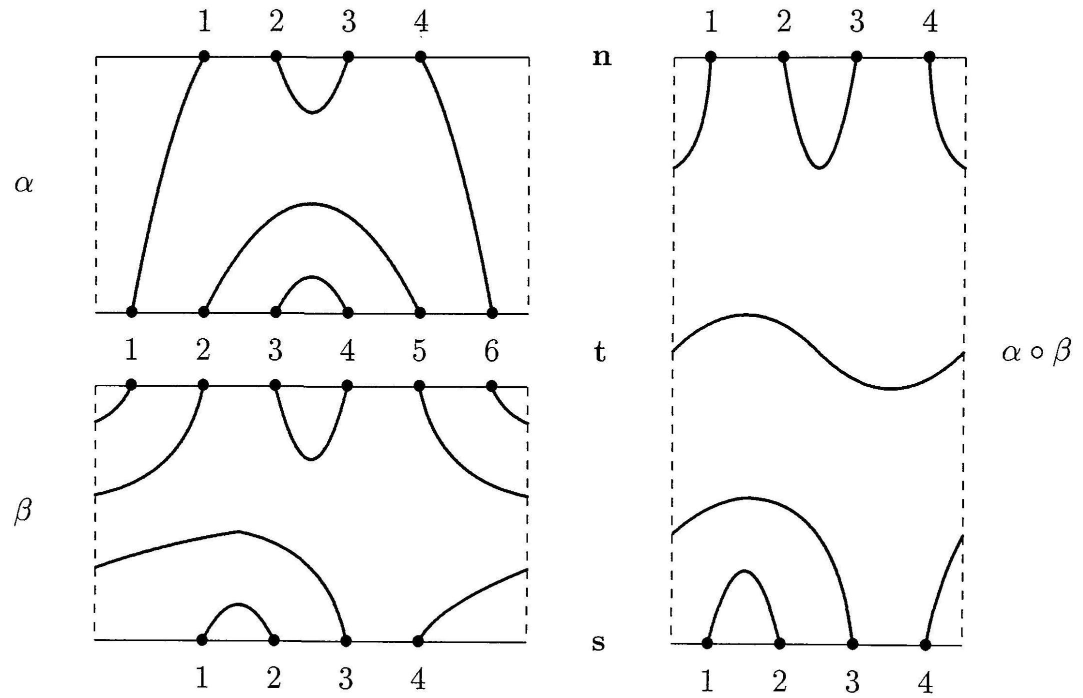
The *rank* $|\alpha|$ of an affine diagram α is the sum of $g(\alpha)$ and the number of vertices $\ell(i,x)$ or $u(i,x)$ with $i < 0$ which are interchanged with vertices $\ell(j,y)$ or $u(j,y)$ with $j \geq 0$. *This is the minimum number of intersections between a graph for* α *and the left side of the fundamental rectangle.*

If an affine diagram α has rank zero, the restriction of the involution ϕ_α to the fundamental rectangle of α is a finite diagram which characterises α; conversely any finite diagram defines a unique affine diagram (by translating its graph). Thus the finite diagrams from t to n may be thought of as special cases of affine diagrams.

Our earlier definition (1.2) of composition for finite diagrams extends to affine diagrams as follows. Let $\alpha\colon t \to n$ and $\beta\colon s \to t$ be affine diagrams. As in the preamble to (1.2), we identify $(\mathbf{Z}\times\mathbf{s})\amalg(\mathbf{Z}\times\mathbf{n}) = \ell(\mathbf{Z}\times\mathbf{s})\cup u(\mathbf{Z}\times\mathbf{n})$ with its image in the disjoint union $(\mathbf{Z}\times\mathbf{s})\amalg(\mathbf{Z}\times\mathbf{t})\amalg(\mathbf{Z}\times\mathbf{n})$, and we extend ϕ_α (resp. ϕ_β) to an involutory bijection $\widetilde{\phi}_\alpha$ (resp. $\widetilde{\phi}_\beta$) of $X = (\mathbf{Z}\times\mathbf{s})\amalg(\mathbf{Z}\times\mathbf{t})\amalg(\mathbf{Z}\times\mathbf{n})$ with fixed point set $\ell(\mathbf{Z}\times\mathbf{s})$ (resp. $u(\mathbf{Z}\times\mathbf{n})$). Denote by H the group of permutations of X which is generated by $\widetilde{\phi}_\alpha$ and $\widetilde{\phi}_\beta$. There is an obvious permutation V of $(\mathbf{Z}\times\mathbf{s})\amalg(\mathbf{Z}\times\mathbf{t})\amalg(\mathbf{Z}\times\mathbf{n})$ whose restrictions to $(\mathbf{Z}\times\mathbf{s})$, $(\mathbf{Z}\times\mathbf{t})$ and $(\mathbf{Z}\times\mathbf{n})$ coincide with the shifts V_s, V_t and V_n of (0.7). This permutation commutes with $\widetilde{\phi}_\alpha$ and $\widetilde{\phi}_\beta$. Therefore V permutes the orbits of H, and since V has finitely many orbits on X, it has finitely many orbits on the set of H-orbits. Let x be this number and let y be the number of H-orbits which are fixed by V. Note that these must be infinite and therefore will correspond to the "horizontal curves" above. By (0.8), we have two types of H-orbits on X. Moreover the loops fall into two types, viz. finite and infinite. These correspond on the cylinder to contractible and incontractible circuits respectively.

(1.4) Definition. Let $\alpha\colon t \to n$ and $\beta\colon s \to t$ be affine diagrams and maintain the above notation. Let $m(\alpha,\beta) := x - y - (s+n)/2$ where x and y are defined in the preamble above. The *composition* $\alpha \circ \beta$ of α and β is the affine diagram $(g(\alpha) + g(\beta) + y,\ \phi_{\alpha\circ\beta})$, where $\phi_{\alpha\circ\beta}$ is the planar involution of $(\mathbf{Z}\times\mathbf{s})\#(\mathbf{Z}\times\mathbf{n})$ which interchanges the ends of arcs (0.8) of H on $(\mathbf{Z}\times\mathbf{s})\amalg(\mathbf{Z}\times\mathbf{t})\amalg(\mathbf{Z}\times\mathbf{n})$ (see above).

A graph for the composition may be obtained in the same way as before by placing a strip with a graph for α above one for β, identifying corresponding points labelled by $\mathbf{Z}\times\mathbf{t}$ and deleting the $m(\alpha,\beta)$ finite (V-orbits of) loops formed. In terms of the corresponding graphs drawn on a cylinder, note that only contractible loops are removed. Interior loops which wrap around the cylinder remain; they correspond to infinite loops in the strip. Here is an illustration.



If $\alpha\colon t\to n$ is a diagram, $\alpha=(g(\alpha),\phi_\alpha)$, its *adjoint* $\alpha^*\colon n\to t$ is given by $\alpha^*=(g(\alpha),\phi_{\alpha^*})$, where ϕ_{α^*} is the planar involution of $(\mathbf{Z}\times\mathbf{n})\#(\mathbf{Z}\times\mathbf{t})$ which interchanges elements $\ell(i,j)$ or $u(p,q)$ with $\ell(i',j')$ or $u(p',q')$ precisely when ϕ_α interchanges $u(i,j)$ or $\ell(p,q)$ with $u(i',j')$ or $\ell(p',q')$. Geometrically, this corresponds to reflecting a graph for α in a horizontal line.

The proof of the following lemma is easy and left to the reader.

(1.5) LEMMA. *Let $\alpha\colon t\to n$, $\beta\colon s\to t$ and $\gamma\colon r\to s$ be (affine) diagrams.*

(1) *The composition $\alpha\circ\beta$ is a diagram $:s\to n$.*

(2) *Composition is associative; i.e. we have $(\alpha\circ\beta)\circ\gamma=\alpha\circ(\beta\circ\gamma)$ and $m(\alpha,\beta)+m(\alpha\circ\beta,\gamma)=m(\beta,\gamma)+m(\alpha,\beta\circ\gamma)$.*

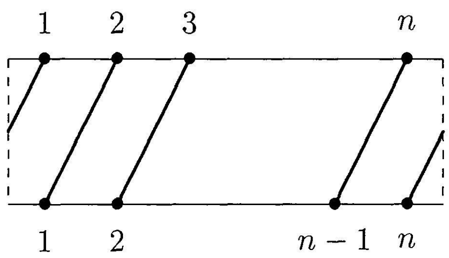
(3) *The finite diagram $id\colon t\to t$ is the identity: $\alpha\circ id=\alpha$ and $id\circ\beta=\beta$.*

(4) *The rank function satisfies $|\alpha\circ\beta|\le|\alpha|+|\beta|$. Both sides of this inequality have the same parity.*

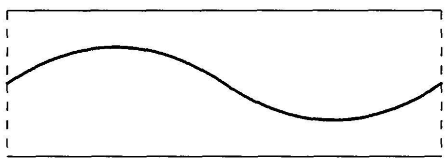
(5) *With the above definition of adjoint, we have $(\alpha\circ\beta)^*=\beta^*\circ\alpha^*$.*

In view of (1.5), we may define the *category* $\mathbf{D}^a$ *of affine diagrams*. This has as objects the non-negative integers and the morphisms from n to m ($n, m \in \mathbf{Z}_{\geq 0}$) are the affine diagrams $\alpha\colon n \to m$. We shall refer to diagrams of even (resp. odd) rank as "even" (resp. "odd").

We now discuss some key examples which play an important rôle in the development below. If σ is any order preserving permutation of $\mathbf{Z} \times \mathbf{n}$, there is a diagram $: n \to n$, also denoted by σ, defined as follows: $\sigma = (0, \phi_\sigma)$, where ϕ_σ is the involution which interchanges lower vertex $\ell(x)$ with upper vertex $u(\sigma(x))$. For example, take $\sigma = \tau_n$ where τ_n is the permutation of $\mathbf{Z} \times \mathbf{n}$ $(n > 0)$ which takes each element to the next largest one. The corresponding diagram $\tau_n\colon n \to n$ appears below. We shall denote by τ_0 the diagram $(1, \phi_{\tau_0})\colon 0 \to 0$ where ϕ_{τ_0} is the unique permutation of the empty set.

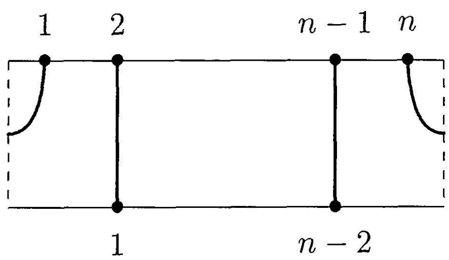


$\tau_n\colon n \to n$

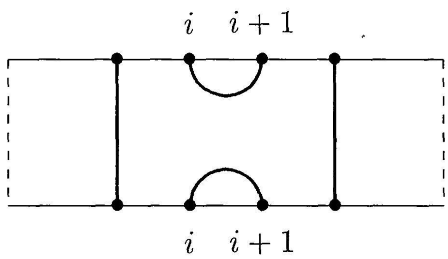


$\tau_0\colon 0 \to 0$

Fix an integer $n \geq 2$. Let ϕ_η be the planar involution of $(\mathbf{Z}\times(\mathbf{n}-\mathbf{2}))\#(\mathbf{Z}\times\mathbf{n})$ defined as follows: ϕ_η interchanges the upper vertices $u(0,n)$ and $u(1,1) = V(u(0,1))$ and the vertices $\ell(0,i)$ and $u(0,i+1)$ for $i = 1, 2, \ldots, n-2$. Let $\eta = \eta_n\colon n-2 \to n$ be the affine diagram $(0, \phi_\eta)$. Define $f_0 = \eta \circ \eta^*$ and $f_i = \tau^i \circ f_0 \circ \tau^{-i}$. Note that the f_j are all diagrams $: n \to n$ and that $f_{i+n} = f_i$. Graphs for these diagrams are depicted below.



$\eta_n\colon n-2 \to n$



$f_i\colon n \to n$

We shall usually use τ and η without the subscript, relying on the context to specify it.

Recall that a morphism $f\colon A \to B$ in any category is *monic* if, for any object X and morphisms $i, j\colon X \to A$ we have $f \circ i = f \circ j \Rightarrow i = j$.

(1.6) LEMMA.

(i) *For any diagram* $\alpha: t \to n$, *the following are equivalent:*

(1) α *is not monic.*

(2) ϕ_α *interchanges some pair of lower vertices.*

(3) $\alpha = \alpha \circ f_i$ *for some* $f_i: t \to t$ *as above.*

(ii) *The monic diagrams* $\sigma: n \to n$ *are precisely the powers* τ_n^i *where* $i \in \mathbf{Z}$ *and* $i \geq 0$ *if* $n = 0$.

Proof. (1) $\Rightarrow$ (2) : If (2) does not hold, then $\alpha^* \circ \alpha$ is the identity $\mathrm{id}: t \to t$ and so α is monic.

(2) $\Rightarrow$ (3) : If $x < \phi_\alpha(x)$ are lower vertices as close as possible, then the planar condition ensures that $\phi_\alpha(x)$ covers x. Thus if i is defined by $x = \ell(0, i)$, then $\alpha = \alpha \circ f_i$.

(3) $\Rightarrow$ (1) : This is immediate.

Part (ii) follows immediately from (i). □

(1.7) DEFINITION. An (affine) diagram $\mu = (g(\mu), \phi_\mu): t \to n$ is *standard* if μ is monic, $g(\mu) = 0$ and ϕ_μ maps each element of $\ell(\{0\} \times \mathbf{t})$ to $u(\{0\} \times \mathbf{n})$.

The *image* of a diagram $\alpha: s \to n$ is the standard diagram constructed as follows. Let $x_1 < x_2 < \cdots < x_t$ be those upper vertices in the fundamental rectangle of $(\mathbf{Z} \times \mathbf{s})\#(\mathbf{Z} \times \mathbf{n})$ which ϕ_α maps to lower vertices and set $t(\alpha) := t$. We refer to $t(\alpha)$ as the number of *through strings* of α. Then the image $i(\alpha): t(\alpha) \to n$ is defined as the monic diagram $i(\alpha) = (0, \phi_{i(\alpha)})$ where $\phi_{i(\alpha)}$ is the involution which interchanges $\ell(0, j)$ with x_j and interchanges upper vertices whenever ϕ_α does. Then any diagram α factors uniquely through its image. Specifically, we have a unique diagram $\rho: s \to t(\alpha)$ such that

$$\alpha = i(\alpha) \circ \rho \quad \text{and} \quad \rho^* \text{ is monic.} \tag{1.7.1}$$

If α is monic then ρ is also monic whence $t(\alpha) = s$ and ρ is a power of τ_s.

A particular case of (1.7.1) which we shall use below relates to the case $s = n$. If $\alpha: n \to n$ is an affine diagram, there are unique integers $t(\alpha)$, $j(\alpha)$ and standard diagrams $\mu, \nu: t(\alpha) \to n$ such that

$$\alpha = \mu \circ \tau_{t(\alpha)}^{j(\alpha)} \circ \nu^* . \tag{1.7.2}$$

(1.8) PROPOSITION. *For any positive integer n, the semigroup generated by the diagrams $f_i\colon n \to n$ is the set of non-monic diagrams $\alpha\colon n \to n$ of even rank.*

Proof. If $\alpha\colon n \to n$ is in the semigroup generated by the f_i, we note that α is even by Lemma 1.5(5) and not monic by the previous lemma.

We prove the converse by induction on length $l(\alpha)$ which is defined by $l(\alpha) = \sum_{i=1}^{n} |\tau^i \circ \alpha \circ \tau^{-i}|$. Let $\alpha\colon n \to n$ be an even and non-monic diagram. Replacing α by $\tau^i \circ \alpha \circ \tau^{-i}$ if necessary, we may assume that $\alpha \circ f_0 = \alpha$, or equivalently that ϕ_α interchanges the lower vertices $\ell(0,1)$ and $\ell(-1,n)$. Since α is even, it follows that $|\alpha| \geq 2$. We shall construct below a diagram $\beta\colon n \to n$ such that $l(\beta) = l(\alpha) - 2$ and $\alpha = \beta \circ f_0$. Assuming that β is not the identity, it is clear that β is even and not monic. By induction β is in the semigroup, and thus so is α.

We now construct β leaving it to the reader to verify that one does obtain a diagram with the properties above. In this proof only, let us say that a vertex $v(i,x)$ (where $v = \ell$ or $v = u$) is negative (for α) if $i < 0$; $v(i,x)$ is special if it is negative and $\phi_\alpha v(i,x)$ is not negative. For example $\ell(-1,n)$ is special.
CASE 1: If $g(\alpha) > 0$ and $\ell(-1,n)$ is the only special lower vertex, let $g(\beta) = g(\alpha) - 1$ and ϕ_β be the involution which interchanges the lower vertices $\ell(i,1)$ and $\ell(i,n)$ (for all $i \in \mathbf{Z}$), and acts as ϕ_α elsewhere.
CASE 2: Otherwise our hypotheses ensure that there is an even number of special vertices. Let y be the minimal special vertex excluding $\ell(-1,n)$. Then let $g(\beta) = g(\alpha)$ and take ϕ_β to be the involution which interchanges $\ell(i,1)$ with $V^i \circ \phi_\alpha(y)$, $\ell(i,n)$ with $V^{i+1}(y)$ (for all $i \in \mathbf{Z}$) and which elsewhere agrees with ϕ_α. $\square$

(1.9) COROLLARY. *If $t < n$ are non-negative integers of the same parity, then the map $\mu \mapsto \mu \circ \eta$ is a bijection between standard diagrams $: t+2 \to n$ and standard diagrams $: t \to n$ of nonzero rank. Here $\eta = \eta_{t+2}\colon t \to t+2$ is the special diagram defined before (1.6) above.*

Proof. The map is well defined and injective, so it suffices to show that it is surjective. If $\nu\colon t \to n$ is a diagram of nonzero rank, then as in the previous proof we may construct $\mu\colon t+2 \to n$ (this is the β of the proof of (1.8)) such that $\mu \circ f_0 = \nu \circ \eta^*$ (this replaces α above). In particular, if ν is standard, then μ is also standard and $\nu = \mu \circ \eta$. $\square$

The result above will be applied later in the following iterated form.

(1.9.1) COROLLARY. *Let $t < s \leq n$ be non-negative integers of the same parity and define k by $s = t+2k$. Write $\eta^k = \eta_s\eta_{s-2}\cdots\eta_{t+4}\eta_{t+2}\colon t \to s$. Then the map $\mu \mapsto \mu \circ \eta^k$ is a bijection between standard diagrams $: s \to n$ and standard diagrams $: t \to n$ of rank $\geq k$. Moreover we have $|\mu \circ \eta^k| = |\mu|+k$.*

The final result of this section provides a method of counting the number of standard diagrams of a given type.

(1.10) DEFINITION. A standard diagram $\mu\colon t \to n$ determines a partition of the set $u(\{0\} \times \mathbf{n})$ of upper vertices into three parts:

$$\begin{aligned}
\mathrm{thr}(\mu) &= \{\phi_\mu(x) \mid x \in \ell(\{0\} \times \mathbf{t})\},\\
\mathrm{rgt}(\mu) &= \{x \in u(\{0\} \times \mathbf{n}) \setminus \mathrm{thr}(\mu) \mid \phi_\mu(x) < x\},\\
\mathrm{lft}(\mu) &= \{x \in u(\{0\} \times \mathbf{n}) \setminus \mathrm{thr}(\mu) \mid \phi_\mu(x) > x\}.
\end{aligned}$$

The names are intended to reflect the facts that any upper vertex either lies on a "through" arc or is the left or right end of an arc between upper vertices. We shall sometimes abuse notation by writing $i \in \mathrm{thr}(\mu)$ if $u(0,i) \in \mathrm{thr}(\mu)$.

(1.11) PROPOSITION. *If n, t and k are non-negative integers such that $n = t+2k$, then the map* lft *induces a bijection between the set of standard diagrams $\mu\colon t \to n$ and subsets of cardinality k of $u(\{0\} \times \mathbf{n})$.*

Proof. We prove by induction on k that a standard diagram $\mu\colon t \to n$ is determined by the set $\mathrm{lft}(\mu)$. The case $k = 0$ is trivial. Replacing μ by a conjugate $\tau_n^i \circ \mu \circ \tau_t^{-j}$ if necessary, we may assume that $\mathrm{lft}(\mu)$ contains $u(0,n)$ but not $u(0,1)$. Since ϕ_μ is planar, the inequality $\phi_\mu(u(1,1)) \leq u(0,n) < u(1,1) \leq \phi_\mu(u(0,n))$ implies $\phi_\mu(u(0,n)) = u(1,1)$. Consequently, $\mu = \eta_n \circ \nu$ where $\nu\colon t \to n-2$ is the standard diagram $\eta_n^* \circ \nu$. By induction $\nu\colon t \to n-2$ is determined by the subset $\mathrm{lft}(\nu) = \{u(0,x-1) \mid u(0,x) \in \mathrm{lft}(\mu),\ x \neq n\}$ and thus $\mu = \eta \circ \nu$ is determined by $\mathrm{lft}(\mu)$. The surjectivity of the map lft is proved in analogous fashion. $\square$

(1.12) COROLLARY. *Let $n, t \in \mathbf{Z}_{\geq 0}$ be integers of the same parity. The number of standard diagrams $\alpha\colon t \to n$ is $\binom{n}{(n-t)/2}$.*

§2. CATEGORIES, ALGEBRAS AND CELL REPRESENTATIONS

In this section we shall define the affine Temperley-Lieb algebras as the sets of endomorphisms in a category $\mathbf{T}^a$ (the "affine Temperley-Lieb category") which is an enrichment of $\mathbf{D}^a$, the category of affine diagrams defined in the last section. We shall construct an uncountable set of representations for these algebras by defining functors from this category to the category of modules over a ring R. It will turn out that these functors provide a "complete set" of representations for the affine Temperley-Lieb algebras. As in the case of diagrams, we shall begin with the finite case.

(2.1) DEFINITION. Let R be a (commutative, associative, unital) ring with an invertible element q. Write $\delta = -(q+q^{-1})$. The *Temperley-Lieb category* $\mathbf{T} = \mathbf{T}_{R,q}$ is defined as follows.

(1) The objects are the non-negative integers.

(2) If $t, n \in \mathbf{Z}_{\geq 0}$, the morphism set $\mathbf{T}(t,n)$ is the free R-module spanned by finite diagrams $: t \longrightarrow n$.

(3) The composition of finite diagrams $\alpha\colon t \longrightarrow n$ and $\beta\colon s \longrightarrow t$ is $\alpha\beta := \delta^{m(\alpha,\beta)}\alpha \circ \beta = (-q-q^{-1})^{m(\alpha,\beta)}\alpha \circ \beta$. Extend bilinearly to define composition in $\mathbf{T}$.

Since q and R will generally be determined by the context, we shall usually suppress them.

The *Temperley-Lieb algebra* $\mathbf{T}(n) = \mathbf{T}(n,n)$ is cellular in the sense of [GL]. It follows that it possesses a family of "cell representations" with canonical bilinear forms. When R is a field, the heads of this family of modules form a complete set of irreducibles for the algebra. Suppose W is any functor $: \mathbf{T} \longrightarrow R$-**mod** from $\mathbf{T}$ to the category R-**mod** of R-modules. Then for $n \in \mathbf{Z}_{\geq 0}$, $W(n)$ is clearly a $\mathbf{T}(n)$-module, so that W provides representations of all the Temperley-Lieb algebras simultaneously. Such functors will therefore be referred to as representations of the category $\mathbf{T}$, or $\mathbf{T}$-modules (see (2.3) below). We show next how the cell modules may be constructed from representations of the category.

(2.2) DEFINITION. Let t be a non-negative integer. The *cell representation* W_t of $\mathbf{T}$ is defined as follows.

(1) For $n \in \mathbf{Z}_{\geq 0}$, $W_t(n)$ is the free R-submodule generated by monic finite diagrams $\mu\colon t \longrightarrow n$.

(2) If $s, n \in \mathbf{Z}_{\geq 0}$ and $\alpha: s \to n$ is a finite diagram, define $W_t(\alpha): W_t(s) \to W_t(n)$ by stipulating that for any finite monic diagram $\mu: t \to s$, $W_t(\alpha)(\mu) = \alpha * \mu$, where

$$\alpha * \mu = \begin{cases} \alpha\mu & \text{if } \alpha \circ \mu \text{ is monic,} \\ 0 & \text{otherwise.} \end{cases}$$

Extend this definition using linearity to obtain the required R-module homomorphism $W_t(\alpha)$.

(3) Let $\langle\ ,\ \rangle_t$ denote the R-bilinear form $W_t(n) \times W_t(n) \to R$ which takes monomorphisms $\mu, \nu: t \to n$ to

$$\langle \mu, \nu \rangle_t = \begin{cases} (-q - q^{-1})^{m(\nu^*, \mu)} & \text{if } \nu^* \circ \mu \text{ is monic,} \\ 0 & \text{otherwise.} \end{cases}$$

(2.3) DEFINITION. A $\mathbf{T}$-module is a functor from the Temperley-Lieb category to the category of R-modules. Parts (1) and (2) of (2.2) define $\mathbf{T}$-modules W_t (for $t \in \mathbf{Z}_{\geq 0}$). The form defined in (3) above is *invariant* in the sense that

$$\langle \alpha * \mu, \nu \rangle_t = \langle \mu, \alpha^* * \nu \rangle_t$$

and so we obtain further $\mathbf{T}$-modules rad_t and L_t where $\mathrm{rad}_t(n)$ is the *radical*

$$\{x \in W_t(n) \mid \langle x, y \rangle_t = 0 \text{ if } y \in W_t(n)\}$$

of this bilinear form and $L_t(n) = W_t(n)/\mathrm{rad}_t(n)$. For $n \in \mathbf{Z}_{\geq 0}$, let $\Lambda(n) = \{t \in \mathbf{Z}_{\geq 0} \mid t \leq n,\ t \equiv n \mod 2\}$ except if $q + q^{-1} = 0$ and n is nonzero and even, in which case we exclude 0 from $\Lambda(n)$. The set $\Lambda(n)$ parametrises the nonzero quotients $L_t(n)$.

(2.4) THEOREM [GL 2.6, 3.2, 3.4]. *Let R be a field and suppose $q \in R$ is nonzero. Let $n \in \mathbf{Z}_{\geq 0}$.*

(1) *If $t \in \mathbf{Z}_{\geq 0}$, M is a $\mathbf{T}(n)$-submodule of the cell module $W_t(n)$ and $s \in \Lambda(n)$ (2.3) is such that there exists a nonzero $\mathbf{T}(n)$-homomorphism $f: W_s(n) \to W_t(n)/M$, then $s \geq t$. If $s = t$, then $f(x) = rx + M$ for some nonzero element r in R.*

(2) *If $s \in \Lambda(n)$, then the radical of $W_s(n)$ as $\mathbf{T}(n)$-module is $\mathrm{rad}_s(n)$, the radical of the form $\langle \mu, \nu \rangle_t$.*

(3) *The family $L_s(n)$ indexed by $s \in \Lambda(n)$ is a complete set of irreducible $\mathbf{T}(n)$-modules.*

We shall now proceed with the affine analogue of (2.4).

(2.5) DEFINITION. Let R be a (commutative, unital) ring with an invertible element q. The *affine Temperley-Lieb category* $\mathbf{T}^a = \mathbf{T}^a_{R,q}$ is defined as follows.

(1) The objects are the non-negative integers.

(2) The morphism set $\mathbf{T}^a(t,n) = \mathbf{T}^a_{R,q}(t,n)$ is the free R-module spanned by the affine diagrams $: t \to n$.

(3) Composition is defined as the R-bilinear map which takes diagrams $\alpha\colon t \to n$ and $\beta\colon s \to t$ to the product $\alpha\beta := \delta^{m(\alpha,\beta)}\alpha \circ \beta = (-q-q^{-1})^{m(\alpha,\beta)}\alpha \circ \beta$, where $m(\alpha,\beta)$ is defined in (1.4).

As in the case of the Temperley-Lieb category, we shall generally omit the subscript (R,q).

We leave it to the reader to check that composition is associative (cf. (1.5)(2)) and that the above definition therefore does make $\mathbf{T}^a$ into a category.

We next define the set which will index the representations of the category $\mathbf{T}^a$ which we shall construct below.

(2.6) DEFINITION. Let Λ^a be the quotient of the set of pairs (t,z) where t is a non-negative integer and z is an invertible element of R by the relation which identifies $(0,z)$ with $(0,z^{-1})$ for all nonzero $z \in R$. Fix $(t,z) \in \Lambda^a$ and define $\chi = \chi_{t,z}\colon \mathbf{T}^a(t,t) \to R$ as the unique R-algebra homomorphism which annihilates non-monic diagrams and is given elsewhere by

$$\begin{aligned} \tau_0{}^i &\mapsto (z+z^{-1})^i \qquad && \text{if } t=0,\\ \tau_t^i &\mapsto z^i && \text{if } t>0. \end{aligned}$$

The (affine) *cell representation* $W_{t,z}$ is the functor from $\mathbf{T}^a$ to R-**mod** defined as follows.

(1) If n is a non-negative integer, $W_{t,z}(n)$ is the R-module generated by monic (affine) diagrams $\mu\colon t \to n$ subject to the relation:

$$\mu \circ \sigma = \chi_{t,z}(\sigma)\mu \qquad \text{if } \sigma\colon t \to t \text{ is monic.}$$

(2) There is an obvious R-bilinear action $\mathbf{T}^a(s,n) \times W_{t,z}(s) \to W_{t,z}(n)$ which takes a diagram $\alpha\colon s \to n$ and monic diagram $\mu\colon t \to s$ to

$$\alpha * \mu = \begin{cases} \alpha\mu & \text{if } \alpha \circ \mu \text{ is monic,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.6.1}$$

Then $\mathbf{T}^a(\alpha)$ is the R-module homomorphism defined by $\mu \mapsto \alpha * \mu$.

The bilinear forms of the finite case are replaced by pairings between related couples of cell modules, which we now define. Let $\langle\ ,\ \rangle_{t,z}$ denote the R-bilinear pairing $W_{t,z}(n) \times W_{t,z^{-1}}(n) \longrightarrow R$ which takes monic diagrams $\mu, \nu \colon t \longrightarrow n$ to

$$\langle \mu, \nu \rangle_{t,z} = \begin{cases} \chi_{t,z}(\nu^* \mu) & \text{if } \nu^* \circ \mu \text{ is monic,} \\ 0 & \text{otherwise.} \end{cases}$$

(2.7) REMARKS. The R-module $W_{t,z}(n)$ is a module for the *affine Temperley-Lieb algebra* $\mathbf{T}^a(n) = \mathbf{T}^a(n, n)$. It has a basis of standard diagrams $: t \longrightarrow n$, because every monic diagram factors uniquely through its image by (1.7.1).

The pairing defined by (2.6.1) is invariant under the $\mathbf{T}^a(n)$ action (see (2.3) for the meaning of invariance). Hence we obtain $\mathbf{T}^a$-modules $\mathrm{rad}_{t,z}$ and $L_{t,z}$ where $\mathrm{rad}_{t,z}(n)$ is the radical in $W_{t,z}(n)$ of this pairing (i.e. the annihilator of $W_{t,z^{-1}}(n)$) and $L_{t,z}(n)$ is $W_{t,z}(n)/\,\mathrm{rad}_{t,z}(n)$.

For $n \in \mathbf{Z}_{\geq 0}$, let $\Lambda^a(n) = \{(t, z) \in \Lambda^a \mid t \leq n,\ t \equiv n \mod 2\}$, with the pair $(0, q)$ ($\equiv (0, q^{-1})$) removed if $q^2 = -1$ and n is nonzero and even. This set parametrises the nonzero $\mathbf{T}^a(n)$-modules $L_{t,z}(n)$. To see this, we have only to show that $\langle\ ,\ \rangle_{t,z} \neq 0$ for $(t, z) \in \Lambda^a(n)$. Write $k = (n-t)/2$ and denote by η^k the standard diagram $\eta_n \eta_{n-2} \dots \eta_{t+4} \eta_{t+2} \colon t \longrightarrow n$. One then verifies easily that

$$\text{(2.7.1)} \qquad \langle \eta^k, \eta^k \rangle = (-q - q^{-1})^k,$$

$$\text{(2.7.2)} \qquad \langle \tau_n \eta^k, \eta^k \rangle = \begin{cases} \chi(\tau_k) = z & \text{if } t > 0, \\ \chi(\tau_0) = z + z^{-1} & \text{if } t = 0, \end{cases}$$

whence the bilinear pairing $\langle\ ,\ \rangle_{t,z}$ is nonzero unless $q^2 = -1$ and $(t, z) = (0, q)$.

(2.8) THEOREM. *Let R be a field with $q \in R$ a nonzero element. Let n be a non-negative integer and $\mathbf{T}^a(n)$ be the affine Temperley-Lieb algebra (2.7).*

(1) *Let $(t, z) \in \Lambda^a$ (see (2.7)), let N be a $\mathbf{T}^a(n)$-submodule of the cell module $W_{t,z}(n)$ (2.6) and take $(s, y) \in \Lambda^a(n)$. Suppose that $f \colon W_{s,y}(n) \longrightarrow W_{t,z}(n)/N$ is a nonzero $\mathbf{T}^a(n)$-homomorphism. Then $s \geq t$. If $s = t$, then $(s, y) = (t, z)$ and $f(x) = rx + N$ for some r in R.*

(2) *For any $(s, y) \in \Lambda^a(n)$, the radical of $W_{s,y}(n)$ as a $\mathbf{T}^a(n)$-module is* $\mathrm{rad}_{s,y}(n)$.

(3) *If R is algebraically closed, then the family $L_{s,y}(n)$ indexed by $(s, y) \in \Lambda^a(n)$ is a complete set of distinct irreducible $\mathbf{T}^a(n)$-modules.*

Proof. The proofs of (1) and (2) are the same as those of [GL 2.6, 3.2, 3.4], given (1.7.2) and recalling that the bilinear forms $\phi_{t,z}$ are non-zero on the modules under consideration. From (1) and (2) it follows that $L_{s,y}(n)$ is an (absolutely) irreducible $\mathbf{T}^a(n)$-module for any $(s,y) \in \Lambda^a(n)$ and that these modules are pairwise inequivalent. Let M be an arbitrary finite dimensional irreducible $\mathbf{T}^a(n)$-module; assuming that $M \neq 0$ (as we may), we shall show that $M \cong L_{t,z}$ for some $(t,z) \in \Lambda^a$. Let $t \in \mathbf{Z}_{\geq 0}$ be minimal such that $\alpha\,.\,m \neq 0$ (. denoting the module action) for some $m \in M$ and $\alpha\colon n \to n$ with t through strings (i.e. $t(\alpha) = t$). Since $M \neq 0$ such t, α and m exist; fix them for the rest of this proof. We shall find an invertible element z in R and construct a nonzero homomorphism $\theta\colon W_{t,z}(n) \to M$. If $q^2 = -1$ and $(t,z) = (0,q)$ for this θ, then α annihilates $W_{t,z}(n)$ when $t(\alpha) = 0$, contradicting our choice of t. Hence if such a θ exists, $(t,z) \in \Lambda^a(n)$. Moreover since M is semisimple, θ factors through its maximal semisimple quotient, which is $L_{t,z}(n)$ by parts (1) and (2). Hence to complete the proof of (3), it suffices to construct the homomorphism θ as above.

Let $\widetilde{W}_t(n)$ denote the free R-module on the set of monic diagrams $\mu\colon t \to n$. There is a $\mathbf{T}^a(n)$-action on $\widetilde{W}_t(n)$ given by

$$\alpha * \mu = \begin{cases} \alpha\mu & \text{if } \alpha \circ \mu \text{ is monic,} \\ 0 & \text{otherwise.} \end{cases}$$

Now $\alpha = i(\alpha) \circ \rho$ for a unique diagram $\rho\colon n \to t$ with ρ^* monic (cf.(1.7.1)). We therefore have a homomorphism $f\colon \widetilde{W}_t(n) \to M$ of $\mathbf{T}^a(n)$-modules given by $f(\mu) = \psi(\mu)\,.\,m$, where $\psi(\mu) = \mu\rho$, with ρ as above. This map is nonzero since $f(i(\alpha)) \neq 0$. Hence f is surjective and there is an element $y \in \widetilde{W}_t(n)$ such that $\psi(y)\,.\,m = m$.

Let $\nu\colon t \to n$ and $\sigma\colon t \to t$ be a pair of monic diagrams. We shall show that

$$\psi(\nu)\psi(y\sigma)\,.\,m = \psi(\nu\sigma)\,.\,m\,. \tag{2.8.1}$$

To see this, observe first that any element x of $\widetilde{W}_t(n)$ may be written uniquely in the form $x = \sum_\mu \mu x_\mu$, where the sum is over the standard diagrams $\mu\colon t \to n$ and the x_μ are in $\widetilde{W}_t(t)$. Write $y = \sum_\mu \mu y_\mu$ accordingly and note that since $\widetilde{W}_t(t)$ is an abelian algebra, if $\rho \circ \mu$ is monic, then $\rho\mu$, y_μ and σ commute with each other. It follows that

$$\begin{aligned}\psi(\nu)\psi(y\sigma)\,.\,m &= \sum_\mu (\nu\rho)(\mu y_\mu \sigma\rho)\,.\,m = \sum_\mu (\nu\sigma)\rho\mu y_\mu\rho\,.\,m \\ &= \psi(\nu\sigma)\psi(y)\,.\,m = \psi(\nu\sigma)\,.\,m,\end{aligned}$$

which proves (2.8.1).

Let σ_1 and σ_2 be monic diagrams from t to t. Taking $\nu = y\sigma_1$ and replacing σ by σ_2 in (2.8.1), we have

(2.8.2) $$\psi(y\sigma_1)\psi(y\sigma_2)\,.\,m = \psi(y\sigma_1\sigma_2)\,.\,m\,.$$

Hence $\widetilde{W}_t(t)$ has an action on the subspace V of M consisting of $\{\psi(y\sigma)\,.\,m \mid \sigma \in \widetilde{W}_t(t)\}$, the element $\sigma \in \widetilde{W}_t(t)$ acting via $\psi(y\sigma)$. Since R is algebraically closed, $\psi(y\tau_t)$ has a nonzero eigenvector $m' = \psi(y\sigma')\,.\,m \in V$, with corresponding eigenvalue ζ (say). Now take $(t, z) \in \Lambda^a$ such that t is as above, $z = \zeta$ if $t > 0$ or $\zeta = z + z^{-1}$ if $t = 0$. Define the character $\chi\colon \widetilde{W}_t(t) \to R$ as in (2.6). Then it follows from (2.8.2) that for any element $\sigma \in \widetilde{W}_t(t)$, we have

$$\psi(y\sigma)\,.\,m' = \chi(\sigma)m'\,.$$

Moreover for any monic diagram $\mu\colon t \to n$, we have

$$\begin{aligned}\psi(\mu\sigma)\,.\,m' &= \psi(\mu\sigma)\psi(y\sigma')\,.\,m = \psi(\mu\sigma\sigma')\,.\,m\\ &= \psi(\mu)\psi(y\sigma\sigma')\,.\,m = \psi(\mu)\psi(y\sigma)\psi(y\sigma')\,.\,m\\ &= \psi(\mu)\psi(y\sigma)\,.\,m'\\ &= \chi(\sigma)\psi(\mu)\,.\,m'\,.\end{aligned}$$

It follows that there is a nonzero homomorphism $\theta\colon W_{t,z}(n) \to M$ of $\mathbf{T}^a(n)$-algebras given by $\theta(\mu) = \psi(\mu)\,.\,m'$. This completes the proof of (2.8). $\square$

The relationship between our affine Temperley-Lieb algebras and the quotient of the Hecke algebra discussed in §0 is explained in the next result.

(2.9) PROPOSITION (cf. [FG]). *There is an algebra homomorphism $\rho\colon H_n^a(q) \to \mathbf{T}^a(n)$ (see (0.1)) which takes T_i to $-f_i - q^{-1}$ for $i = 1, \ldots, n$. The kernel of ρ is the ideal I_n^a of (0.5), while the image of ρ is spanned by non-monic diagrams : $n \to n$ of even rank, together with the identity. Thus the latter diagrams span an algebra which is isomorphic to $TL_n^a(\delta)$ (see (0.5)).*

Idea of proof. Write $C_i = -(T_i + q^{-1})$. Then $H_n^a(q)$ is generated as R-algebra by $C_1, \ldots, C_n$ subject to the relations

$$C_iC_j = C_jC_i \quad \text{if } |i-j| \geq 2 \text{ and } (i,j) \neq (1,n)$$
$$C_i^2 = \delta C_i$$
$$C_iC_{i+1}C_i - C_i = C_{i+1}C_iC_{i+1} - C_{i+1} = -q^{-3}E_i\,, \text{ where } E_i \text{ is as in §0}$$

and the index i is taken modulo n in the last equation. One checks easily that the diagrams f_i satisfy these relations with E_i replaced by 0, whence the indicated map defines a homomorphism of algebras. The kernel contains the E_i, and hence contains I_n^a. The image is the algebra generated by the f_i, which is identified as in the statement of (2.9) by Corollary (1.9). To see that the kernel is no larger than I_n^a, we refer the reader to [FG]. □

The modules $W_{t,z}(n)$ and $L_{t,z}(n)$ may be regarded as modules for the subalgebra TL_n^a of $\mathbf{T}^a(n)$. It is a simple consequence of (2.9) that as TL_n^a-modules $W_{t,z} \cong W_{t,y}$ if $t = n$ or $z + y = 0$. Moreover the argument of [GL 2.6] shows that $L_{t,z}$ remains irreducible as a TL_n^a-module, unless $t = 0$ and $z^2 = -1$. In the remaining case, $W_{0,z}$ is the direct sum of two submodules $W_{0,z}^+$ and $W_{0,z}^-$ spanned respectively by the even and odd standard diagrams $: 0 \to n$. If $q^2 \neq -1$ these modules have irreducible heads $L_{0,z}^+$ and $L_{0,z}^-$ whose sum is $L_{0,z}$. This leads to the following description of the cell modules and irreducible modules for the algebra TL_n^a.

(2.9.1) COROLLARY. *Let $\overline{\Lambda}^a(n)$ be the quotient of $\Lambda^a(n)$ by the equivalence relation $(t,z) \equiv (t,y)$ if $t = n$ or $z = -y$, with new points $(0,z)^+$ and $(0,z)^-$ replacing $(0,z) \in \Lambda^a(n)$ if n is even and $z^2 = -1$. Then Theorem (2.8) applies to $W_{t,z}(n)$ and $L_{t,z}(n)$ regarded as TL_n^a-modules, with $\overline{\Lambda}^a(n)$ replacing $\Lambda^a(n)$ and the representations being realised as above.*

(2.10) THE JONES ANNULAR ALGEBRAS

The *Brauer centraliser algebra* is the free R-module $\mathbf{B}(n)$ generated by fixed point free (but not necessarily planar) involutions ϕ of $\mathbf{n}\#\mathbf{n}$ with multiplication defined analogously to (1.2) and (2.1). There is a unique algebra homomorphism: ψ: $\mathbf{T}^a(n) \to \mathbf{B}(n)$ which takes an (affine) diagram $\alpha: n \to n$ to $(\delta)^{g(\alpha)}$ times the involution ψ_α of $\mathbf{n}\#\mathbf{n}$ which interchanges the vertex $\ell(x)$ or $u(x)$ with $\ell(y)$ or $u(y)$ $(x, y \in \mathbf{n})$ precisely when ϕ_α maps $\ell(i,x)$ or $u(i,x)$ to $\ell(j,y)$ or $u(j,y)$ for some $i,j \in \mathbf{Z}$. *Jones' annular algebra* $\mathbf{J}(n)$ is the image of this algebra homomorphism. This algebra is known to have a cellular structure [GL]; the associated cell modules are related to those of $\mathbf{T}^a$ as follows. If $(t,z) \in \Lambda^a(n)$ is such that $t > 0$ and $z^t = 1$, then the kernel of ψ annihilates the $\mathbf{T}^a(n)$-module $W_{t,z}(n)$, and so we obtain $\mathbf{J}(n)$-modules $W_{t,z}(n)$ and $L_{t,z}(n)$; with the notation of [GL] the first module is (canonically isomorphic to) the cell representation $W(t,z)$ while the second is its unique irreducible head $L(t,z)$. The remaining cell representation $W(0,1)$ of $\mathbf{J}(n)$ as defined in [GL], is the quotient $W_{0,q}(n)/M$ where M is the image of the map θ_n: $W_{2,1}(n) \to W_{0,q}(n)$ of Theorem (3.4) below. The unique head $L(0,1)$ of $W(0,1)$ is $L_{0,q}(n)$.

It is therefore clear that the representation theory of the Jones algebra $J(n)$ is included in the representation theory of our affine algebras. Its cell representations form a subset of those of $\mathbf{T}^a(n)$, with one exception.

The (finite) Temperley-Lieb category $\mathbf{T}$ is a subcategory of the affine Temperley-Lieb category $\mathbf{T}^a$. Therefore the cell representations of $\mathbf{T}^a$ give rise to representations of $\mathbf{T}$ by restriction. We complete this section by describing the structure of the resulting restricted $\mathbf{T}(n)$ modules, as well as some "asymptotic" ones.

(2.11) LEMMA. *Let R be a ring with an invertible element q. Consider the affine Temperley-Lieb category $\mathbf{T}^a = \mathbf{T}^a_{R[z,z^{-1}],q}$ over the ring $R[z,z^{-1}]$ of Laurent polynomials in an indeterminate z and let $t \in \mathbf{Z}_{\geq 0}$. Define coefficient functions $r^\mu_\nu(x) \in R[z,z^{-1}]$ of the cell representation $W_{t,z}$ by:*

$$x * \mu = \sum_{\substack{\nu:\, t \to s \\ \text{standard}}} r^\mu_\nu(x)\nu \quad \text{in } W_{t,z}(s)$$

where $\mu\colon t \to n$ and $\nu\colon t \to s$ are standard (affine) diagrams and $x \in \mathbf{T}^a(n,s)$. If x is a finite diagram α, then the coefficient $r^\mu_\nu(\alpha)$ vanishes unless $l = |\mu| - |\nu| \geq 0$ and $\nu \circ \tau^i_t = \alpha \circ \mu$ for some $i \in \mathbf{Z}$. In this case $r^\mu_\nu(\alpha)z^l$ is a polynomial in $R[z^2]$ of degree at most l. Furthermore if μ_1 and μ_2 are standard diagrams from t to n, $\langle \mu_1, \mu_2 \rangle_{t,z} z^{|\mu_1|+|\mu_2|}$ also lies in $R[z^2]$.

Proof. Although these statements are staightforward consequences of Lemma (1.5), we provide the details for the reader's convenience. Recall that $\alpha * \mu$ is equal to $\alpha \circ \mu$ if this has t through strings, and is zero otherwise. In the former case, $\alpha \circ \mu = \nu \circ \tau^j_t$ for some standard ν and $j \in \mathbf{Z}$, and we have

$$\alpha * \mu = \delta^{m(\alpha,\mu)} y^j \nu$$

where $y = z$ if $t > 0$ and $y = z + z^{-1}$ if $t = 0$. Thus the coefficient $r^\mu_\nu(\alpha)$ vanishes except for this particular standard diagram ν, and $r^\mu_\nu(\alpha) = \delta^{m(\alpha,\mu)} y^j$.

We now relate the ranks to j. Since $|\alpha| = 0$ by (1.5) we have $|\alpha \circ \mu| \leq |\mu|$ and these have the same parity. Since ν is standard, we also have $|\nu \circ \tau^j| = |j| + |\nu|$ whence $|j| \leq |\mu| - |\nu| = l$ and still both sides have the same parity. Since the left hand side is nonnegative, so is the right hand side. Furthermore, $r^\mu_\nu(\alpha)$ is an R-linear combination of integer powers of z, all of which have the parity of j, the smallest of which is $-j$ and the largest being j. It follows that $r^\mu_\nu(\alpha)z^l$ is a polynomial in z^2, of degree $(j+l)/2 \leq l$, which proves the first statement.

Next we compute the bilinear pairing. Let μ_1 and $\mu_2 : t \to n$ be standard. Their scalar product vanishes, unless $\mu_2^* \circ \mu_1 : t \to t$ has t through strings. In this case we have $\mu_2^* \circ \mu_1 = \tau_t^k$ for some $k \in \mathbf{Z}$ and

$$\langle \mu_1, \mu_2 \rangle_{t,z} = \delta^{m(\mu_2^*, \mu_1)} y^k$$

where y is as above. The proof is now completed as above, bearing in mind that by (1.5) $|k| \leq |\mu_1| + |\mu_2|$ and the two sides have the same parity. □

As a consequence of Lemma (2.11), we may construct $\mathbf{T}$-modules $W_{t,0}$ and $W_{t,\infty}$ as follows. If $n \in \mathbf{Z}_{\geq 0}$, let $W_{t,0}(n)$ be the free R-module generated by standard affine diagrams (see (1.7)) $\mu : t \to n$. If $\alpha : n \to s$ is a finite diagram and $\mu : t \to n$ is standard, define

(2.11.1) $$\alpha * \mu = r_\nu^\mu(\alpha)_0 \nu$$

where ν is as in (2.11) and $r_\nu^\mu(\alpha)_0$ is the constant term of $r_\nu^\mu(\alpha) z^l$. Extend this R-bilinearly to an action of $\mathbf{T}(n)$. The module $W_{t,\infty}$ is constructed in analogous fashion using the constant term of $r_\nu^\mu(\alpha) z^{-l}$ in $R[z^{-2}]$. One then has an invariant pairing

(2.11.2) $$\langle\ ,\ \rangle_{t,0} : W_{t,0}(n) \times W_{t,\infty}(n) \to R$$

where $\langle \mu, \nu \rangle_{t,0}$ is the constant term of $\langle \mu, \nu \rangle_{t,z} z^{|\nu| + |\mu|}$.

(2.12) THE FINITE TEMPERLEY-LIEB ALGEBRAS

We shall describe how the affine modules are related to the finite modules of [GL,§6] (see (2.2) above). Let $t \in \mathbf{Z}_{\geq 0}$ and let z be 0, ∞, or an invertible element of R. We construct a filtration of the $\mathbf{T}$-module $W_{t,z}$ whose quotients are cell representations. If $s \in \mathbf{Z}_{\geq 0}$ is such that $s \equiv t \mod 2$, then for each $n \in \mathbf{Z}_{\geq 0}$ let $W_{t,z}^s(n)$ be the R-span of the standard affine diagrams $\mu : t \to n$ of rank $|\mu| < (s-t)/2$. This defines an increasing family

$$0 = W_{t,z}^t(n) \subset W_{t,z}^{t+2}(n) = W_t(n) \subset W_{t,z}^{t+4}(n) \subset \cdots \subset W_{t,z}^n(n) \subset W_{t,z}^{n+2}(n) = W_{t,z}(n)$$

of $\mathbf{T}$-submodules of $W_{t,z}(n)$ (of course $W_{t,z}^s(n) = 0$ for $s \leq t$ and $W_{t,z}^s(n) = W_{t,z}(n)$ for $s > n$). It follows from (1.9.1) that there is an exact sequence of natural transformations:

(2.12.1) $$0 \longrightarrow W_{t,z}^s \longrightarrow W_{t,z}^{s+2} \longrightarrow W_s \longrightarrow 0$$

where the left map is inclusion and the right map is given (cf. (1.9.1)) at n by:

$$W_{t,z}^{s+2}(n) \to W_s(n) : \mu \circ \eta^{(s-t)/2} \mapsto \mu .$$

(2.13) THE TRIVIAL REPRESENTATION OF THE FINITE TEMPERLEY-LIEB ALGEBRAS

The cell module $W_s(s)$ is one-dimensional and will be referred to as the *trivial representation* of $\mathbf{T}(s)$. Observe that the diagrams $f_i \in \mathbf{T}(s)$ all act as the zero operator in this representation, whence if e_s is the corresponding idempotent in $\mathbf{T}(s)$ (e_s exists generically, by generic semisimplicity), then $f_i * e_s = 0 = e_s * f_i$ for all i. The idempotent e_s is referred to in the literature (cf. [MV], [We], [Li] and [J3], where e_s was first identified) as the Jones, or augmentation idempotent of $\mathbf{T}(s)$.

(2.14) LEMMA. *Let t, s and k be non-negative integers such that $s = t + 2k$. If $x \in W_{t,z}(s)$ is annihilated by all finite diagrams $\alpha: s \to s$ except id_s, then x is a scalar multiple of $e_s * \eta^k$, where e_s is defined above and η^k is defined in (1.9.1).*

Proof. We may suppose that $k > 0$, since the case $k = 0$ is trivial. The hypothesis implies that Rx is a realization of the trivial representation of $\mathbf{T}(s)$, whence $x \in e_s * W_{t,z}(s)$. We shall therefore be done if we show that

$$e_s * W_{t,z}(s) = Re_s * \eta^k . \tag{2.14.1}$$

Now η^k is characterised among the standard diagrams $: t \to s$ as the unique diagram of maximal rank (k). If $\mu: t \to s$ is standard and $|\mu| < k$, then $\mu = f_i * \nu$, for some standard diagram $\nu: t \to s$ and $i \in \{1, 2, \ldots, s-1\}$ because ϕ_μ must interchange two upper vertices in the fundamental rectangle (recall $k > 0$). Hence $e_s * \mu = e_s * f_i * \nu = 0$, proving (2.14.1) and hence the lemma. □

§3. HOMOMORPHISMS AND NATURAL TRANSFORMATIONS

For any integer n, define the Gaussian integer $[n]_{\mathrm{x}}$ in the function field $\mathbf{Q}(\mathrm{x})$ by

$$[n]_{\mathrm{x}} := \frac{\mathrm{x}^n - \mathrm{x}^{-n}}{\mathrm{x} - \mathrm{x}^{-1}} = \mathrm{x}^{n-1} + \mathrm{x}^{n-3} + \cdots + \mathrm{x}^{1-n} .$$

Define the Gaussian x-factorial by

$$[n!]_{\mathrm{x}} = [n]_{\mathrm{x}}[n-1]_{\mathrm{x}} \ldots [2]_{\mathrm{x}}[1]_{\mathrm{x}} .$$

For any pair $n \geq k$ of positive integers, the Gaussian binomial coefficient is

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\mathrm{x}} = \frac{[n]_{\mathrm{x}}[n-1]_{\mathrm{x}} \dots [n-k+1]_{\mathrm{x}}}{[k]_{\mathrm{x}}[k-1]_{\mathrm{x}} \dots [1]_{\mathrm{x}}} \quad \text{and} \quad \begin{bmatrix} n \\ 0 \end{bmatrix}_{\mathrm{x}} = 1 .$$

These are Laurent polynomials in x, so that we may speak of $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q$ for any invertible element q of a ring R. If q^2 has finite order $l > 1$, then $[n]_q = 0$ iff l divides n.

(3.1) DEFINITION. A *forest* is a partially ordered set such that if $x \le y$ and $x \le z$ then $y \le z$ or $z \le y$.

(3.2) EXAMPLES.

(1) The set of orbits of a planar involution ϕ (of a totally ordered set) is a forest $F(\phi)$ with order defined by: $X \le Y$ if X is contained in the convex hull of Y. For a finite diagram $\alpha\colon t \to n$, the associated forest for ϕ_α is denoted $F(\alpha)$.

(2) For any affine diagram $\mu\colon t \to n$, order the set $\mathrm{lft}(\mu)$ (see (1.10)) by stipulating that $y \preceq x$ if $x \le y \le \phi_\mu(x)$ or $x \le Vy \le \phi_\mu(x)$. This condition amounts to the requirement that the convex hull of the orbit of x contains some translate of the orbit of y. The resulting poset is a forest which we denote by $P(\mu)$.

The following result is well known.

(3.3) PROPOSITION (Stanley [RS]). *Let P be a forest of cardinality n; for $y \in P$ denote by h_y the number of elements of P which are less than or equal to y. Then the rational function*

$$h_P(\mathrm{x}) := \frac{[n!]_{\mathrm{x}}}{\prod_{y \in P} [h_y]_{\mathrm{x}}}$$

is a Laurent polynomial with integer coefficients.

It is possible to strengthen the proofs of [RS (5.3) and (22.1)] to yield that the coefficients of $h_P(\mathrm{x})$ are actually positive, but we do not require this here.

The next result, one of the main ones in this work, provides the homomorphisms between cell modules which enable us to analyse them. For any affine diagram $\mu\colon t \to s$, we sometimes (e.g. in the statement of (3.4)) identify the set $u(\{0\} \times \mathrm{s})$ with s in the obvious way, thereby identifying the sets $\mathrm{thr}(\mu)$, $\mathrm{rgt}(\mu)$ and $\mathrm{lft}(\mu)$ with subsets of s.

(3.4) THEOREM. *Let R be a ring with an invertible element q. Let t, s and k be non-negative integers such that $t+2k=s$. Let $z \in R$ be such that $z^2 = q^s$ and set $y = zq^{-k}$, so that $y^2 = q^t$. Then there exists a natural transformation $\theta\colon W_{s,y} \to W_{t,z}$ of $\mathbf{T}^a$-modules (2.5, 2.6) whose component at n applied to a monic diagram $\nu\colon s \to n$ is given by:*

(3.4.1)
$$\theta_n(\nu) = \sum_{\substack{\mu\colon t \to s \\ \text{standard}}} q^i z^{k-|\mu|} h_{P(\mu)}(q)\, \nu\mu$$

where $2i = s(|\mu|-k) + t(s+1)/2 - \sum_{u(x,0)\in \mathrm{thr}(\mu)} x$, $|\alpha|$ is the rank (see (1.3) et seq.) of the affine diagram α and $h_{P(\mu)}(\mathrm{x})$ is the polynomial associated to the forest $P(\mu)$ of (3.2)(2) by (3.3).

Proof. We shall assume without loss of generality that R is the function field $\mathbf{Q}(q^{1/2})$ and that $s > t$, the case $s = t$ having been covered in Theorem (2.8).

To define a natural transformation θ from $W_{s,y}$ to $W_{t,z}$, we require, for each $n \in \mathbf{Z}_{\geq 0}$, a homomorphism $\theta_n\colon W_{s,y}(n) \to W_{t,z}(n)$ such that for any pair n, m of non-negative integers and diagram $\alpha\colon n \to m$, the following diagram commutes:

(3.4.2)
$$\begin{tikzcd} W_{s,y}(n) \arrow[r, "W_{s,y}(\alpha)"] \arrow[d, "\theta_n"'] & W_{s,y}(m) \arrow[d, "\theta_m"] \\ W_{t,z}(n) \arrow[r, "W_{t,z}(\alpha)"] & W_{t,z}(m) \end{tikzcd}$$

Now $W_{s,y}(s)$ is a one dimensional R-module with basis $\mathrm{id}_s\colon s \to s$. Write $\theta_s(\mathrm{id}_s) = \mathbf{v}$. Taking $n = m = s$ and $\alpha = \tau_s$ in (3.4.2) we see that

(1)
$$\tau_s * \mathbf{v} = y\mathbf{v}.$$

Moreover if we take $m = s-2$ and $n = s$, then $W_{s,y}(m) = 0$ whence $\alpha * \mathbf{v} = 0$ for any $\alpha \in \mathbf{T}^a(s, s-2)$. In particular, taking $\alpha = \eta_s^*$, we obtain

(2)
$$\eta_s^* * \mathbf{v} = 0.$$

It follows that (1) and (2) are necessary conditions for the particular θ of (3.4.1) to define a natural transformation. We shall prove (1) and (2) shortly, but first show that they are sufficient for the proof of the theorem. Suppose $\mathbf{v} \in W_{t,z}(s)$ is such that (1) and (2) hold.

Then $\mathbf{v} = \sum_{\substack{\mu\colon t\to s\\ \text{standard}}} c_\mu(q,z)\mu$ for certain coefficients $c_\mu(q,z) \in R$. Define θ by

(3.4.3)
$$\theta_n(\nu) = \sum_{\substack{\mu\colon t\to s\\ \text{standard}}} c_\mu(q,z)\nu\mu$$

for any monic diagram $\nu\colon s \to n$. Then (1) implies that the formula (3.4.3) for θ_n defines a unique R-linear map $W_{s,y}(n) \to W_{t,z}(n)$ for each n; these maps are clearly $\mathbf{T}^a(n)$-module homomorphisms. In order to prove that this family of maps defines a natural transformation, fix a diagram $\alpha\colon n \to m$ and standard diagram $\nu\colon s \to n$. If $\alpha \circ \nu$ is also monic, then

$$\begin{aligned}\alpha * \theta_n(\nu) = \alpha * (\nu * \mathbf{v}) = (\alpha\nu) * \mathbf{v} &= (-q-q^{-1})^{m(\alpha,\nu)}(\alpha\circ\nu) * \mathbf{v}\\ &= (-q-q^{-1})^{m(\alpha,\nu)}\theta_m(\alpha\circ\nu) = \theta_m(\alpha * \nu).\end{aligned}$$

On the other hand, if $\alpha\circ\nu$ is not monic, then by Lemma 1.6 there is $i \in \mathbf{Z}_{\geq 0}$ such that $\alpha\circ\nu = \alpha\circ\nu\circ f_i = \beta\circ\eta^*\circ\tau^{-i}$ where $\beta = \alpha\circ\nu\circ\tau^i$; hence we have

$$\alpha * \theta_n(\nu) = (-q-q^{-1})^{m(\alpha,\nu)}(\alpha\circ\nu) * \mathbf{v} = (-q-q^{-1})^{m(\alpha,\nu)}y^{-i}\beta * \eta^* * \mathbf{v} = 0$$

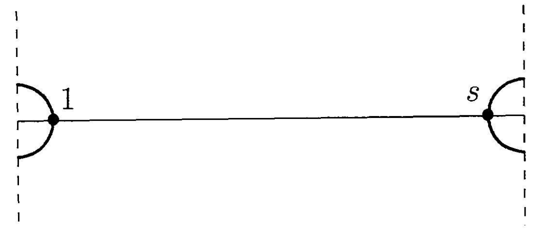
while $\theta_m(\alpha * \nu) = \theta_m(0) = 0$, proving that the squares (3.4.2) commute. It follows that θ is a natural transformation if (1) and (2) hold.

We therefore turn to the proof of (1) and (2) for the particular $\mathbf{v}$ defined by (3.4.1). First we establish (1). Let $\mu\colon t \to s$ be standard, let ν be the image of $\tau\circ\mu$ and recall that $\tau\circ\mu = \nu\circ\sigma$ for some monic diagram $\sigma\colon t \to t$. Suppose first that $s \notin \operatorname{thr}(\mu)$; then σ is the identity. Using the abuse of notation explained after (1.10), we have $\operatorname{thr}(\nu) = \{x+1 \mid x \in \operatorname{thr}(\mu)\}$ and ϕ_ν agrees with $\tau_s\circ\phi_\mu\circ\tau_s^{-1}$ elsewhere. Hence $\sum_{x\in\operatorname{thr}(\nu)} x = t + \sum_{x\in\operatorname{thr}(\mu)} x$, $|\nu| = |\mu| \pm 1$ and $h_{P(\nu)}(\mathrm{x}) = h_{P(\mu)}(\mathrm{x})$. Alternatively, assume that $s \in \operatorname{thr}(\mu)$. Then $t > 0$ and $\sigma = \tau$. We have $\operatorname{thr}(\nu) = \{x+1 \mid x \in \operatorname{thr}(\mu), x \neq s\} \cup \{1\}$ and ϕ_ν agrees with $\tau\circ\phi_\mu\circ\tau^{-1}$ elsewhere. Hence $\sum_{x\in\operatorname{thr}(\nu)} x = t - s + \sum_{x\in\operatorname{thr}(\mu)} x$, $|\nu| = |\mu|$ and $h_{P(\nu)}(\mathrm{x}) = h_{P(\mu)}(\mathrm{x})$. In either case, the coefficient of ν in $y\mathbf{v}$ equals the coefficient of ν in $\tau * \mathbf{v}$ and (1) follows.

To complete the proof of the theorem, it remains only to prove (2). Fix a standard affine diagram $\nu\colon t \to s-2$. We consider standard diagrams $\mu\colon t \to s$ such that ν is the image of $\eta^*\circ\mu$, because these index the terms in the expression (3.4.1) which contribute to the coefficient of ν in $\eta^* * \mathbf{v}$; we shall show that the sum of these contributions is zero. In the figures below, we depict the upper edge of the fundamental rectangle of μ.

Let h denote the coefficient of $\eta\circ\nu$ in $\mathbf{v}$. Let $a' \in \mathbf{s}$ be minimal subject to $a' > 1$ and $\phi_{\eta\circ\nu}(u(0,a')) \notin u(\{0\}\times\mathbf{s})$. Similarly, let $b' \in \mathbf{s}$ be maximal subject to $b' < s$ and $\phi_{\eta\circ\nu}(u(0,b')) \notin u(\{0\} \times \mathbf{s})$. Define $a = a'/2$ and $b = (s+1-b')/2$. We shall consider four types of diagrams μ and compute the contribution of each type to the coefficient of ν in $\eta^* * \mathbf{v}$ separately. Note that the stipulation that ν is the image of $\eta^* \circ \mu$ implies that μ is determined completely by the images $\phi_\mu(u(0,1))$ and $\phi_\mu(u(0,s))$.

CASE 1: $\phi_\mu(u(0,1)) = u(-1,s)$.

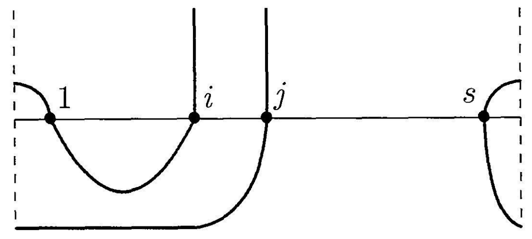


It follows that $\mathrm{thr}(\mu) = \mathrm{thr}(\nu)$ and $\mu = \eta \circ \nu$. Thus $\eta^* * \mu = (-q-q^{-1})\nu$ and so the contribution of the term μ to the coefficient of ν is

$$(-q-q^{-1})h = -[2]_q h.$$

CASE 2: Suppose $\phi_\mu(u(-1,s)) > u(0,1)$.

Then $u(0,s) \notin \mathrm{thr}(\mu)$ by planarity. If $\phi_\mu(u(-1,s)) = u(0,j)$ (with $j > 1$), then $\phi_\mu(u(0,1)) = u(0,i)$ and clearly $i < j < a'$ since ϕ_μ is planar.



Since $\eta^* * \mu = \nu$, μ contributes its own coefficient in $\mathbf{v}$ to the coefficient of ν in $\eta^* * \mathbf{v}$. It is easily checked that this coefficient may be expressed as

$$\frac{[1]_q[(j-i+1)/2]_q}{[(j+1)/2]_q[i/2]_q} h.$$

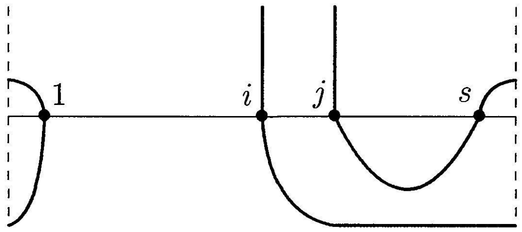
Now the interval $u(0,2), u(0,3), \ldots, u(0,a'-1)$ is a union of $\phi_{\eta\circ\nu}$-orbits. These form a subforest Q of the forest of (3.2)(1) and in this subforest, the $\phi_{\eta\circ\nu}$-orbit $(u(0,i), u(0,j))$ is clearly maximal. Moreover there is an obvious bijection between the maximal orbits $u(0,i_r), u(0,j_r)$ $(r = 1, \ldots, l)$ of $\phi_{\eta\circ\nu}$ on $u(0,2), u(0,3), \ldots, u(0,a'-1)$ and the diagrams μ satisfying the condition $\phi_\mu(u(-1,s)) > u(0,1)$ under which $\phi_\mu(u(-1,s)) = u(0,j_r)$. If (i_1,j_1), (i_2,j_2),..., (i_l,j_l) are the possible pairs (i_r,j_r) as above, listed in order of increasing i_r, then $i_1 = 2$, $j_l = a' - 1$ and $j_k + 1 = i_{k+1}$ for $k = 1, \ldots, l-1$. A straightforward induction argument shows that altogether this family contributes

$$\frac{[a-1]_q}{[a]_q} h$$

to the coefficient of ν.

CASE 3: $\phi_\mu(0,1) < (-1,s)$ and $1 \notin \mathrm{thr}(\mu)$.

If $i = \phi_\mu(1,1)$ and $j = \phi_\mu(s)$, then $b' < i < j$ since ϕ_μ is planar.

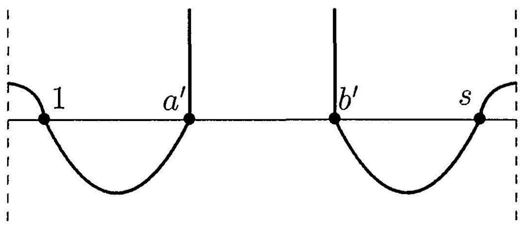


This case is the mirror image of case 2, working from the right instead of the left. Arguing as above, one finds that the total contribution from the μ of this type is

$$\frac{[b-1]_q}{[b]_q} h .$$

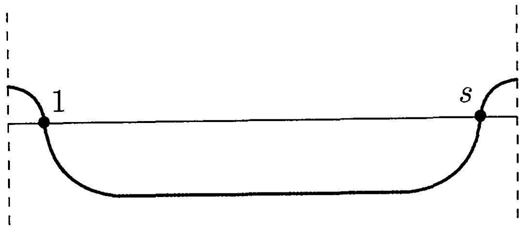
CASE 4: Otherwise.

We shall see that there are just one or two remaining diagrams. First assume that the rank of ν is nonzero. Then it follows from the planar nature of $\eta \circ \nu$ and the choice of a' and b' that $\phi_\eta \circ \nu$ interchanges $u(0,a')$ and $u(i,b')$ for some $i \in \mathbf{Z}$. Since μ is planar, $u(0,1), u(0,s) \notin \mathrm{thr}(\mu)$, so that $\phi_\mu(u(0,1)) > u(0,1)$ and $\phi_\mu(u(0,s)) < u(0,s)$ and it follows that $\phi_\mu(u(0,1)) = u(0,a')$ and $\phi_\mu(u(0,s)) = u(0,b')$.



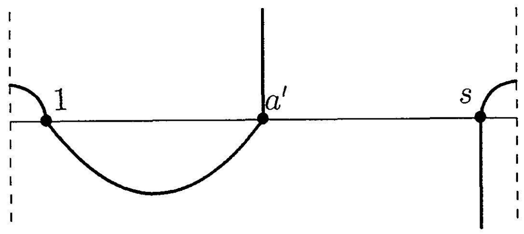
One now computes that this diagram μ contributes $([a+b]_q/([a]_q[b]_q))h$ to the coefficient of ν.

Alternatively, assume that ν has rank zero, i.e. is finite. If $\mathrm{thr}(\nu)$ is empty (in which case $t = 0$), then $1, s \notin \mathrm{thr}(\nu)$ (being empty) and it follows that $\mu(u(0,1)) = u(0,s)$.

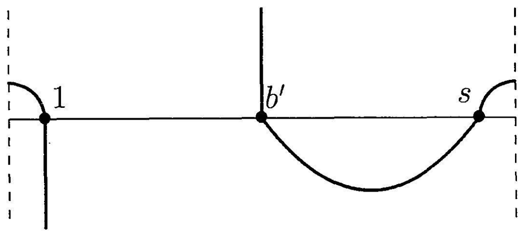


This diagram μ contributes $\chi(\tau_0)q^{-s/2}z^1([1]_q/[s/2]_q)h = ([a+b]_q/([a]_q[b]_q))h$ to the coefficient of ν.

Finally assume that $\mathrm{thr}(\nu)$ is nonempty (and ν is finite). Then $u(0, a'-1)$ and $u(0, b'-1)$ are the minimum and maximum elements of $\mathrm{thr}(\nu)$ respectively. Equivalently, $u(0, a')$ and $u(0, b')$ are the minimum and maximum elements of $\mathrm{thr}(\eta \circ \nu)$ respectively. It follows that either $u(0, s) \in \mathrm{thr}(\mu)$ (in which case $\phi_\mu(u(0,1)) = u(0, a')$)



or $u(0,1) \in \mathrm{thr}(\mu)$ (in which case $\phi_\mu(u(0,s)) = u(0,b')$).



Together these two diagrams contribute

$$\chi(\tau)q^{a-s}z^1\frac{[1]_q}{[a]_q}h + \chi(\tau^{-1})q^{-b}z^1\frac{[1]_q}{[b]_q}h = \frac{[a+b]_q}{[a]_q[b]_q}h\,.$$

We may now compute the sum of the contributions from all four cases:

$$\left(-[2]_q + \frac{[a-1]_q}{[a]_q} + \frac{[b-1]_q}{[b]_q} + \frac{[a+b]_q}{[a]_q[b]_q}\right)h = 0\,.$$

Thus the coefficient of ν in $\eta^* * \mathbf{v}$ vanishes and (2) follows. $\square$

We next prove the following consequence of Theorem (3.4) for the cell modules of the finite Temperley-Lieb algebras. In discussing these, we think of $\mathbf{t}\#\mathbf{n}$ as the fundamental rectangle $\ell(\{0\}\times\mathbf{t}) \cup u(\{0\}\times\mathbf{n})$ of $(\mathbf{Z}\times\mathbf{t})\#(\mathbf{Z}\times\mathbf{n})$ as explained in the discussion after (1.3).

(3.5) COROLLARY. *Let R be a field with an invertible element q. Let t, s be non-negative integers of the same parity such that $t \leq s$. Then there is a natural transformation $\theta\colon W_s \to W_{t,\infty}$ of $\mathbf{T}$-modules (see (2.2) and (2.11.1)) whose component at n applied to a finite monic diagram $\nu\colon s \to n$ is given by*

$$\theta_n(\nu) = \sum_{\substack{\mu\colon t \to s \\ \text{standard}}} h^f_\mu(q)\nu\mu \tag{3.5.1}$$

where $h^f_\mu(x)$ is the polynomial associated in (3.3) to the subforest of $F(\phi_\mu)$ formed by the orbits of ϕ_μ which intersect the fundamental rectangle **t#n** *non-trivially.*

Proof. This result may be established by a computation similar to the one above. However we shall deduce it from Theorem (3.4). First we give a different construction for $W_{t,\infty}$. Let t and s be as above and choose $l', k' \in \mathbf{Z}_{\geq 0}$ such that $k' + l' > s$ and $l' = t + k'$. Set $m = l' + s + k'$ and $z = q^{m/2}$. Define an embedding $\bar{\ }\colon \mathbf{T}(s) \to \mathbf{T}^a(m)$ by mapping f_i to $\bar{f}_i := f_{i+l'}$ for $i = 1, 2, \dots, s-1$. We say that a monic diagram $\mu\colon 0 \to m$ is *distinguished* if $|\mu| = 0$ and the involution ϕ_μ does not interchange two elements of $u(\mathbf{l}')$ or two elements of $u(\{m, m-1, \dots, m-k'+1\})$. There is a one to one correspondence ψ between distinguished diagrams $\mu\colon 0 \to m$ and standard diagrams $\nu\colon t \to s$; μ corresponds to ν when ϕ_ν interchanges vertices $u(i)$ and $u(j)$ in $u(\mathbf{s})$ iff ϕ_μ interchanges $u(i + l')$ with $u(j + l')$ in $u(\mathbf{m})$. This defines μ completely, since $\operatorname{lft}(\mu)$ contains $u(\mathbf{l}')$, so that $\operatorname{lft}(\mu)$ is determined, whence μ is, by (1.11).

Suppose $\alpha\colon s \to s$ is finite and $\mu\colon 0 \to m$ is standard. Then $\bar{\alpha} \circ \mu$ is distinguished $(\colon 0 \to m)$ only if μ is distinguished. Hence the R-submodule M of $W_{0,z}(m)$ spanned by the non-distinguished standard diagrams $\mu\colon 0 \to m$ is invariant under $\mathbf{T}(s)$. The $\mathbf{T}(s)$-module $W_{0,z}(m)/M$ has basis $\mu + M$ indexed by distinguished diagrams $\mu\colon 0 \to m$, which may be identified using the map ψ above with the standard diagrams $\colon t \to s$. This identification may be extended R-linearly to an isomorphism $\psi\colon W_{0,z}(m)/M \to W_{t,\infty}(s)$ of $\mathbf{T}(s)$-modules.

Now Theorem (3.4) provides an explicit natural transformation $\theta\colon W_{m,1} \to W_{0,z}$. The image $\theta_m(\mathrm{id}_m) = \mathbf{v}$ is given by (3.4.1). Let $\mathbf{w} = \theta^f_s(\mathrm{id}_s)$; i.e. $\mathbf{w}$ is the right hand side of (3.5.1) with $\nu = \mathrm{id}_s$. Then it is easily checked that the isomorphism ψ takes $\mathbf{v} + M$ to $\mathbf{w}$. It follows that $\mathbf{w}$ is annihilated by any non-monic finite diagram and consequently, by an argument similar to that which follows (3.4.2), that the family $\{\theta_n\}$ of homomorphisms given by (3.5.1) defines a natural transformation between the functors W_s and $W_{t,\infty}$. $\square$

(3.6) COROLLARY. *In addition to the hypotheses of the previous corollary, assume that q^2 has finite order $l > 1$. If $t < s < t + 2l$ and $s + t \equiv -2$* mod $2l$, *then there is a natural transformation $\theta\colon W_s \to W_t$ of* $\mathbf{T}$*-modules whose component at n is*

$$\theta_n^f(\nu) = \sum_{\substack{\mu\colon t \to s \\ \text{monic} \\ \text{finite}}} h_{F(\mu)}(q)\nu\mu \tag{3.6.1}$$

where $h_{F(\mu)}(\mathrm{x})$ is the polynomial of (3.3) for the forest of (3.2)(1) associated to the planar involution ϕ_μ.

Proof. Let $\mu\colon t \to s$ be a monic affine diagram and consider the forest A of orbits of ϕ_μ which intersect $\mathbf{t\#s}$ non-trivially, as in (3.5). Let B be the ideal of A generated by those ϕ_μ-orbits which contain a lower vertex, and let $C = A \setminus B$. If $x \in B$ and $y \in C$, then $x \not\geq y$ and $x \not\leq y$. It follows that

$$h_A(\mathrm{x}) = h_B(\mathrm{x})h_C(\mathrm{x})\begin{bmatrix} a \\ c \end{bmatrix}_{\mathrm{x}}$$

where $h_A(\mathrm{x})$, $h_B(\mathrm{x})$ and $h_C(\mathrm{x})$ are the Laurent polynomials associated by Proposition (3.3) to the forests A, B and C of cardinality a, b and $c = a - b$ respectively. Since $c \leq (s - t)/2 < l$, the denominator $[r!]_{\mathrm{x}}$ of the Gaussian binomial coefficient does not vanish when we set x equal to q and we have

$$\begin{bmatrix} a \\ c \end{bmatrix}_q = \frac{[a]_q \ldots [b+1]_q}{[c]_q \ldots [1]_q}.$$

If μ has nonzero rank, then $a = (t + s)/2 + |\mu| \geq (t + s)/2 + 1 > b$. Since $2l$ divides $s + t + 2$, the numerator vanishes and so $h_{F(\mu)}(q) = h_A(q) = 0$.

Thus the image $\theta_s^f(\mathrm{id})$ of (3.5.1) actually lies in the submodule $W_{t,\infty}^{t+2}(s)$ which is canonically isomorphic to $W_t(s)$. Therefore the right side of (3.6.1) (with $n = s$ and $\nu = \mathrm{id}_s$) is annihilated by non-monic diagrams, and so the argument following (3.4.2) shows that (3.6.1) defines a natural transformation. □

The next result gives an explicit closed formula for the "Jones" or "augmentation" idempotent in the singular case, i.e. when q is a root of unity. There are recursive [We] and partial results concerning formulae for this idempotent, but to our knowledge, the closed formula we give below is new (see also [Li]).

(3.7) COROLLARY. *Assume that q^2 has (multiplicative) order l in R. Then the primitive idempotent (sometimes referred to as the Jones or augmentation idempotent) $e \in \mathbf{T}(l-1)$ which is associated with the trivial representation of $\mathbf{T}(l-1)$ is given by*

$$e = \sum_{\alpha} h_{F(\alpha)}(q)\alpha$$

where the sum is over finite diagrams $\alpha\colon l-1 \to l-1$ and $h_{F(\alpha)}(x)$ is the polynomial associated to the forest of orbits of ϕ_α.

Proof. It clearly suffices to prove that for any non-identity finite diagram $\beta\colon l-1 \to l-1$, we have $\beta * e = e * \beta = 0$. Now the finite diagrams $\alpha\colon l-1 \to l-1$ are in canonical bijection with finite diagrams $\alpha'\colon 0 \to 2l-2$; to see this, imagine the line of lower vertices of α rotated clockwise until it is collinear with the line of upper vertices of α, giving a graph for α'. Moreover if α, β are two finite diagrams $: l-1 \to l-1$, it is easily verified that

$$(\alpha\beta)' = \beta^* \circ \alpha' \tag{3.7.1}$$

where $\beta^* \in \mathbf{T}(l-1)$, regarded as a subalgebra of $\mathbf{T}(2l-2)$ in the usual way i.e. as the subalgebra generated by $\{f_1, \ldots, f_{l-2}\} \in \mathbf{T}(2l-2)$. By (3.6), there is a homomorphism $\theta\colon W_{2l-2}(2l-2) \to W_0(2l-2)$ with image the R-span of $e' := \sum_{\substack{\alpha'\colon 0\to 2l-2\\ \text{finite}}} h_{F(\alpha')}(q)\alpha'$. But under the identification above, $h_{F(\alpha)}(q) = h_{F(\alpha')}(q)$ for any finite diagram $\alpha\colon l-1 \to l-1$. Hence under the identification, e' corresponds to the element e of the statement. But $\mathbf{T}(l-1)$ clearly acts on this image via the trivial representation. By (3.7.1), it follows that $\mathbf{T}(l-1)$ acts on Re via the trivial representation as required. □

(3.8) REMARK. Part of the significance of (3.7) derives from the fact that the element e is known to generate the radical of Jones' trace function [J1] on the Temperley-Lieb algebra $\mathbf{T}(N)$ (for any N), $\mathbf{T}(l-1)$ being regarded as a subalgebra of $\mathbf{T}(N)$ as explained in the proof of (3.7) and therefore yields a presentation of Jones' projection algebra.

More specifically, Jones (op. cit.) showed that there is a unique trace $tr\colon \mathbf{T}(N) \to R$ which satisfies $tr(1) = 1$ and $tr(xf_i) = \delta^{-1}tr(x)$ for any element $x \in \mathbf{T}(i) \subseteq \mathbf{T}(i+1)$. This trace defines an Hermitian (or bilinear) form on $\mathbf{T}(N)$, which is known to be degenerate if and only if $l \leq N+1$, i.e. $N \geq l-1$. When the Jones form is degenerate, the element $e \in TL(l-1)$ generates (as ideal of $\mathbf{T}(N)$) its radical. Jones' projection algebra $A_{N,\beta}$ [J1] is defined as the quotient of $\mathbf{T}(N)$ by this ideal; hence we obtain an presentation for $A_{N,\beta}$ by simply adding the relation $e = 0$ to the usual presentation of the Temperley-Lieb algebra. For a discussion of other contexts for e, see [MV].

We remark also that it follows from (3.6) (cf. also §5 below) and the theory of cellular algebras that $\mathbf{T}(N)$ is non-semisimple if and only if $N \geq l$. Thus the case $N = l - 1$ is distinguished as the unique one where $\mathbf{T}(N)$ is semisimple, but the Jones form is degenerate.

(3.9) REMARK concerning the Jones (annular) algebras. Since the Jones algebra $\mathbf{J}(n)$ (see (2.10) above) is a quotient of the algebra $\mathbf{T}^a(n)$, any $\mathbf{J}(n)$-module lifts to a $\mathbf{T}^a(n)$-module. The $W_{t,z}(n)$ which correspond to $\mathbf{J}(n)$-modules in this way are those where $z^t = 1$ and $t > 0$ (2.10). Now the conditions $z^2 = q^s$ and $y = zq^{-k}$ (where $s = t + 2k$) of Theorem (3.4) imply (if $t > 0$) that $z^t = 1$ if and only if $y^s = 1$. Hence if $z^t = 1$, the modules $W_{t,z}(n)$ and $W_{s,y}(n)$ of (3.4) may be thought of as $\mathbf{J}(n)$-modules and the map θ_n as a homomorphism of $\mathbf{J}(n)$-modules. If $t = 0$, $z = q$ and the order l of q^2 is finite, then Theorem (3.4) provides a homomorphism $W_{s,y} \to W_{0,q}/M : x \mapsto x + M$ where $s = 2l - 2$, $y = q^l (= \pm 1)$ and M is the module defined in (2.9).

§4. DISCRIMINANTS

(4.1) DEFINITION. Throughout this section R denotes the function field $\mathbf{Q}(q)$ and we consider the affine Temperley-Lieb algebras over the ring $R[z, z^{-1}]$ of Laurent polynomials. If $t \leq s$ are non-negative integers of the same parity define

$$[t; s]_{\mathrm{X}} := \begin{bmatrix} s \\ (s-t)/2 \end{bmatrix}_{\mathrm{X}} .$$

The goal of this section is to compute the *discriminant* of the bilinear pairing

$$\langle\ ,\ \rangle_{t,z} : W^s_{t,z}(n) \times W^s_{t,z^{-1}}(n) \to R \quad (n \in \mathbf{Z}_{\geq 0}) .$$

This is the determinant of the *gram matrix* $G^s_{t,z}(n)$ with entries $\langle \mu, \nu \rangle_{t,z}$ indexed by pairs of standard monic diagrams $: t \to n$ of rank (strictly) less than $(s-t)/2$. Recall from (2.12) that these diagrams span a $\mathbf{T}(n)$-submodule $W^s_{t,z}(n)$ of $W_{t,z}(n)$ and that these submodules form an increasing filtration of $W_{t,z}(n)$ as s increases. When $n < s$, we write $G_{t,z}(n)$ for this matrix, because it is then independent of s. Similarly define the gram matrix $G^s_{t,0}(n)$ for the pairing $\langle\ ,\ \rangle_{t,0} : W^s_{t,0}(n) \times W^s_{t,\infty}(n) \to R$ and let $G_t(n)$ denote the gram matrix of $\langle\ ,\ \rangle_t : W_t(n) \times W_t(n) \to R$ with respect to the basis of finite, monic diagrams. We maintain the standard notation $s - t = 2k$.

Recall from (2.13) that there is an idempotent $e_s \in \mathbf{T}(s)$ associated with the trivial representation $W_s(s)$. Define the element $\mathbf{v}_s \in W_{t,z}(s)$ by

$$\mathbf{v}_s := [t;s]_q e_s * \eta^k = \sum_{\substack{\mu:\ t\to s \\ \text{standard}}} e_\mu \mu .$$

Note that by (2.14) $\mathbf{v}_s$ spans the projection of $W_{t,z}(s)$ onto the trivial representation of $\mathbf{T}(s)$. We conjecture, but do not require, that the coefficients (in $R = \mathbf{Q}(q)$) of the Laurent polynomials e_μ in q actually lie in $\mathbf{Z}_{\geq 0}[q, q^{-1}]$.

(4.2) PROPOSITION. *With the notation above,*

$$\text{(4.2.1)} \qquad \left\langle \mathbf{v}_s, \eta^k \right\rangle_{t,z} = \prod_{\substack{t<r\leq s \\ r\equiv t \mod 2}} (z^2 - q^r - q^{-r} + z^{-2}) .$$

Proof. By Lemma (2.11), $e_\mu z^l$ is a polynomial in $R[z^2]$ of degree at most $l = k - |\mu|$. We shall use Theorem (3.4) to compute the value of e_μ when z^2 is specialised to q^s. Taking $n = m = s$ in (3.4.1), we see that $\theta_s(\mathrm{id}_s)$ is annihilated by finite non-monic diagrams $\alpha\colon s \to s$. It follows from (2.14) that $\theta_s(\mathrm{id}_s)$ is a scalar multiple of the specialisation of $\mathbf{v}_s$. The coefficient of η^k in $\theta_s(\mathrm{id}_s)$ is easily checked from the formula (3.4.1) to be 1. Since $e_s * e_s * \eta^k = e_s * \eta^k$, we see that the coefficient of η^k in $e_s * \eta^k$ is also 1, whence after specialisation, we have $[t;s]_q \theta_s(\mathrm{id}_s) = \mathbf{v}_s$. Hence e_μ specialises to

$$\text{(4.2.2)} \qquad q^i z^{k-|\mu|} h_{P(\mu)}(q)[t;s]_q .$$

Similarly, Corollary (3.5) shows that the coefficient of z^l in e_μ is

$$\text{(4.2.3)} \qquad h_{F(\mu)}(q)$$

where $h_{F(\mu)}(\mathrm{x})$ is as defined in (3.5).

Now the statement (4.2) will follow by induction on s from the claim:

$$\text{(4.2.4)} \qquad \eta^* * \mathbf{v}_s = (z^2 - q^s - q^{-s} + z^{-2})\mathbf{v}_{s-2} .$$

We now proceed to establish (4.2.4), using the observations just made. If α is a finite diagram in $\mathbf{T}(s-2)$, then $\eta \circ \alpha = \beta \circ \eta$ for some finite diagram $\beta\colon s \to s$. If α is not the identity, then β is not monic and so α^* annihilates $\eta^* * \mathbf{v}_s$. It follows from (2.14) that

$$\eta^* * \mathbf{v}_s = \lambda \mathbf{v}_{s-2}$$

for some scalar λ in $R = \mathbf{Q}(q)$.

To determine this scalar, we compute the coefficient of η^{k-1} in $\eta^* * \mathbf{v}_s$ and compare this with the corresponding coefficient $[t; s-2]_q$ in $\mathbf{v}_{s-2}$. In the proof of Theorem (3.4), we enumerated the standard diagrams $\mu: t \to s$ such that $\eta^* \circ \mu = \nu = \eta^{k-1}$. We now compute the contribution of each such μ to the coefficient of η^{k-1} in $\eta^* * \mathbf{v}_s$, just as in the proof of (3.4).

In case 1, $\mu = \eta^k$ and the contribution is

$$-[2]_q[t; s]_q$$

to the coefficient of ν. Cases 2 and 3 do not arise because $a' = 2$ and $b' = s-1$. There are three possibilities that arise in Case 4. Suppose first that μ has nonzero rank, or equivalently that $s-2 > t$; hence $\phi_\mu(1) = 2$ and $\phi_\mu(s) = s-1$ as in the proof of (3.4). It follows that $|\nu| - |\mu| = 2$ and so e_μ has the form $r_2 z^2 + r_0 + r_{-2} z^{-2}$ for some r_2, r_0, r_{-2} in $R = \mathbf{Q}(q)$. We have $r_{-2} = r_2$ by symmetry, $r_2 = [t; s-2]_q$ by (4.2.3) and $q^s r_2 + r_0 + q^{-s} r_{-2} = [2]_q[t; s]_q$ by (4.2.2). Thus the contribution of μ is

$$(4.2.5) \qquad (z^2 - q^s - q^{-s} + z^{-2})[t; s-2]_q + [2]_q[t; s]_q .$$

Otherwise we may assume that μ is finite, or equivalently that $t = s-2$. If $t = 0$, then $\phi_\mu(1) = 2 = s$ and the coefficient e_μ has the form $r_1 z + r_{-1} z^{-1}$ for some $r_1, r_{-1} \in R$. We have $r_1 = r_{-1}$ by symmetry and $r_1 = 1$ by (4.2.3). Hence this term contributes $\chi(\tau_0)(z + z^{-1})$ which is equal to the expression in (4.2.5). Next suppose $t = s-2 > 0$. Then either $s \in \operatorname{thr}(\mu)$ and $\phi_\mu(1) = 2$, or $1 \in \operatorname{thr}(\mu)$ and $\phi_\mu(s) = s-1$. In the first case $e_\mu = r_1 z + r_{-1} z^{-1}$ and by symmetry in the second case the coefficient is $r_{-1} z + r_1 z^{-1}$. We have $r_1 = 1$ and $r_{-1} = [s-1]_q$ by (4.2.3). Hence these terms contribute $\chi(\tau)(z + [s-1]_q z^{-1}) + \chi(\tau^{-1})([s-1]_q z + z^{-1})$ which is also equal to the expression (4.2.5).

Each of the three possibilities yields the same contribution (4.2.5), from which it follows that the coefficient of ν in $\eta^* * \mathbf{v}_s$ is $\lambda[t; s-2]_q$ where $\lambda = z^2 - q^s - q^{-s} + z^{-2}$. The claim (4.2.4), and hence the proposition, follows. $\square$

(4.3) COROLLARY. *For non-negative integers $t \leq s$ of the same parity, we have the recurrence:*

$$\det G^{s+2}_{t,z}(n) = \det G^s_{t,z}(n) \det G_s(n) \left([t; s]_q^{-1} \prod_{\substack{t < r \leq s \\ r \equiv t \bmod 2}} (z^2 - q^r - q^{-r} + z^{-2}) \right)^{\dim W_s(n)}$$

where $n \in \mathbf{Z}_{\geq 0}$. This, together with the initial condition $\det G^t_{t,z}(n) = 1$ determines $\det G^s_{t,z}(n)$ for any n, s, t.

Proof. Define a basis of $W_{t,z}^{s+2}(n)$ as follows. If $\mu\colon t \to n$ has rank (strictly) less than $k = (s-t)/2$, define $\mathbf{v}_\mu = \mu$. Alternatively, if $\mu\colon t \to n$ has rank k, then (1.9.1) shows that there exists a unique finite monic diagram $\mu'\colon s \to n$ such that $\mu = \mu' \circ \eta^k$; define $\mathbf{v}_\mu = \mu' * \mathbf{v}_s$ and note that $\mathbf{v}_\mu = [t;s]_q \mu$ mod $W_{t,z}^s(n)$. The discriminant of the pairing $\langle -,- \rangle_{t,z}$ with respect to this basis is therefore

(4.3.1) $$[t;s]_q^{2\dim W_s(n)} \det G_{t,z}^{s+2}(n)\,.$$

We obtain the recurrence above by computing this discriminant in another way.

If $\mu\colon t \to s$ is standard and $\mu \neq \eta^k$, then

(4.3.2) $$\langle \mathbf{v}_s, \mu \rangle_{t,z} = 0\,.$$

Together with the previous proposition, this implies that for any finite diagram $\alpha\colon s \to s$,

(4.3.3) $$\langle \alpha * \mathbf{v}_s, \mathbf{v}_s \rangle_{t,z} = \begin{cases} [t;s]_q \lambda & \text{if } \alpha = \mathrm{id}\,, \\ 0 & \text{otherwise,} \end{cases}$$

where $\lambda = \langle \mathbf{v}_s, \eta^k \rangle_{t,z}$, which is given explicitly in (4.2.1).

Let $\mu, \nu\colon t \to n$ be standard of rank at most k. If $|\mu| = k$ and $|\nu| < k$, then

$$\langle \mathbf{v}_\mu, \mathbf{v}_\nu \rangle_{t,z} = \langle \mathbf{v}_s, (\mu')^* * \nu \rangle_{t,z} = 0$$

by (4.3.2). If $|\mu| < k$ and $|\nu| < k$, then $\langle \mathbf{v}_\mu, \mathbf{v}_\nu \rangle_{t,z} = \langle \mu, \nu \rangle_{t,z}$. If $|\mu| = |\nu| = k$, then $\langle \mu', \nu' \rangle_t$ is the coefficient of the identity in $\nu'^* \nu'$ and so (4.3.3) shows that

$$\langle \mathbf{v}_\mu, \mathbf{v}_\nu \rangle_{t,z} = [t;s]_q \lambda \langle \mu', \nu' \rangle_t\,.$$

Therefore the discriminant of the pairing on $W_{t,z}^{s+2}(n)$ with respect to this basis is

$$\det G_{t,z}^s(n) \times \det G_s(n) \left([t;s]_q \lambda\right)^{\dim W_s(n)},$$

which, taking account of (4.3.1) above, completes the proof of (4.3). □

(4.4) COROLLARY. *With the notation above,*

(4.4.1) $$\det G_{t,z}^s(n) = \det G_{t,0}^s(n) \prod_{\substack{t<r<s \\ r \equiv t \mod 2}} (z^2 - q^r - q^{-r} + z^{-2})^{\dim W_{r,z}^s(n)}\,.$$

Proof. Comparing the coefficients of the highest power of z^{-1} on both sides of (4.3) we see that

(4.4.2) $$\det G_{t,0}^{s+2}(n) = \det G_{t,0}^{s}(n) \det G_s(n)[t;s]_q^{-\dim W_s(n)} .$$

If we write $Q(s) = \det G_{t,z}^{s}(n)/\det G_{t,0}^{s}(n)$, then it follows from (4.3) and (4.4.2) that

$$Q(s+2) = Q(s) \prod_{\substack{t<r\leq s \\ r\equiv t \mod 2}} (z^2 - q^r - q^{-r} + z^{-2})^{\dim W_s(n)} .$$

This recurrence for $Q(s)$ is easily solved using the fact that $Q(t) = 1$. Taking into account the relation $\dim W_{t,z}^{s+2} = \dim W_t(n) + \dim W_{t+2}(n) + \cdots + \dim W_s(n)$, which is an easy consequence of (2.12.1), the desired equation (4.4.1) follows. □

(4.5) COROLLARY. *With the notation above,*

$$\det G_{t,0}^{s}(n) = \frac{\det G_{t,0}(n)}{\det G_{s,0}(n)} \prod_{\substack{r\geq s \\ r\equiv t \mod 2}} \left(\frac{[t;r]_q}{[s;r]_q}\right)^{\dim W_r(n)} .$$

Proof. The recurrence (4.4.2) shows that

(4.5.1) $$\det G_{t,0}(n) = \det G_{t,0}^{s}(n) \prod_{\substack{r\geq s \\ r\equiv t \mod 2}} \det G_r(n)[t;r]_q^{-\dim W_r(n)} .$$

For (4.4.2) to hold for all $s \geq t$, we must take $\det G_{s,0}^{s}(n)$ to be equal to 1. Hence

$$\prod_{\substack{r\geq s \\ r\equiv t \mod 2}} \det G_r(n) = \det G_{s,0}(n) \prod_{\substack{r\geq s \\ r\equiv t \mod 2}} [s;r]_q^{\dim W_r(n)} .$$

Substituting this into (4.5.1), we obtain the statement. □

(4.6) PROPOSITION. *If $t \leq n$ are non-negative integers of the same parity, then*

$$\det G_{t,0}(n) = \pm 1 .$$

Proof. Identify (as above) $\mathbf{n}$ with $u(\{0\}\times \mathbf{n})$. Let $k = (n-t)/2$. Partially order the set of cardinality-k subsets of $\mathbf{n}$ as follows: if $x_1 < x_2 < \cdots < x_k$ and $y_1 < y_2 < \cdots < y_k$ are sequences of elements of $\mathbf{n}$, we say that $\{x_i\} \leq \{y_i\}$ if $x_j \leq y_j$ for all j in $\mathbf{k}$.

We claim that if $\mu, \nu : t \to n$ are standard, then $\langle \mu, \nu \rangle_{t,0} = 0$ unless $\mathrm{rgt}(\mu) \geq \mathrm{lft}(\nu)$. Furthermore if $\mathrm{rgt}(\mu) = \mathrm{lft}(\nu)$, then $\langle \mu, \nu \rangle_{t,0} = 1$. That is, the gram matrix with respect to this pair of ordered bases is triangular with diagonal entries all equal to one, whence its determinant is one. Hence the result will follow from these two claims.

Let $\mu, \nu\colon t \to n$ be standard. Choose graphs for μ and ν with the property that each edge crosses the left side of the fundamental rectangle at most once and recall from section one the construction of a graph for the composition $\sigma = \nu^* \circ \mu$. First suppose that $\langle \mu, \nu \rangle_{t,0} \neq 0$; then $\sigma = \tau^{-|\mu|-|\nu|}\colon t \to t$ since $z = 0$ (cf. (2.11.2)). In this case it is possible to orient the edges of the graphs of μ and ν in such a way that:

(1) Each lower vertex of μ is a source.

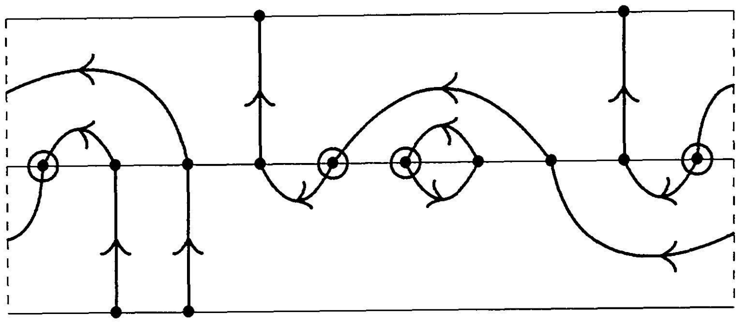
(2) Each lower vertex of ν is a sink.

(3) Each upper vertex $x \in u(\mathbf{Z} \times \mathbf{n})$ is a source (or sink) in precisely one of μ and ν.

(4) Each edge of μ or ν which crosses the left side of the fundamental rectangle is directed from right to left; that is if $x, y \in u(\{0\} \times \mathbf{n})$ are such that $\phi_\mu(x) = V(y)$ (resp. $\phi_\nu(x) = V(y)$) then this edge is directed from y to x in the graph of μ (resp. ν).

To see this, observe that the property (4) implies that when the graphs of μ and ν^* are juxtaposed to form the composition $\nu^* \circ \mu$, the orientations of their edges match, giving an orientation (i.e. linear ordering) to the $\langle \widetilde{\phi}_\mu, \widetilde{\phi}_{\nu^*} \rangle$-orbits on $(\mathbf{Z} \times \mathbf{t}) \amalg (\mathbf{Z} \times \mathbf{n}) \amalg (\mathbf{Z} \times \mathbf{t})$ which are described in the preamble to (1.4). Conversely, such an ordering on these orbits gives an orientation with the required properties. We therefore describe such an ordering or orientation on the orbits which will satisfy the above requirements. If $t = 0$, orient the $g(\sigma) = |\mu| + |\nu|$ infinite loops (see preamble to (1.4) – these correspond to incontractible circuits on the cylinder) from right to left. If $t > 0$, orient each edge of σ ("through string") from the lower vertex to the upper vertex. Note that since $|\nu^* \circ \mu| = |\nu^*| + |\mu|$, all edges of the graphs of μ and ν^* which cross the left side of the fundamental rectangle are included in the edges of the graph of σ, i.e. lie on the through strings of the composite graph. Thus only the contractible (finite) loops which are contained in the fundamental rectangle remain and these may be oriented arbitrarily (say, anti-clockwise). The properties (1) to (4) are clear. Moreover it is also easy to see that if such an orientation exists, then $\sigma = \tau^{-|\mu|-|\nu|}$, since the conditions imply that $|\nu^* \circ \mu| = |\nu^*| + |\mu|$. An example is depicted in the diagram opposite.

Let a denote the i-th element of $\mathrm{rgt}(\mu)$, where $\mathbf{n}$ is identified with $u(\{0\} \times \mathbf{n})$, etc. Now there are at least i sources of the (directed



EXAMPLE

The sources of μ in $\mathbf{n}$ are circled

graph of) μ in the interval $\mathbf{a} \subseteq \mathbf{n}$, because when $y \in \mathrm{rgt}(\mu)$ and $y \le a$, there is precisely one source in the set $\{y, \phi_\mu(y)\} \cap \mathbf{a}$, by property (4) above. Similarly, let b denote the i-th element of $\mathrm{lft}(\nu)$. Then the above argument shows that there are at least $k - i$ sinks of ν in $\{b+1, b+2, \ldots, n\} \subseteq \mathbf{n}$ and since, by property (2), ν has k sinks in $\mathbf{n}$, there are at most i sinks of ν in $\mathbf{b}$. Moreover if the number of sinks of ν in $\mathbf{b}$ is precisely i, any arc of ν from $b \in \mathbf{n}$ to an element of $\{b+1, b+2, \ldots, n\} \subseteq \mathbf{n}$ must have sink b, otherwise the number of sinks of ν in $\{b+1, b+2, \ldots, n\} \subseteq \mathbf{n}$ would be greater than $k - i$. Now by property (3), a sink of ν is a source of μ. Hence if $b > a$, the number of sources of μ which $\le b$ is i,so that by the argument just given, μ is a sink of ν, hence a source of μ. But the number of sources of μ which $\le a$ is $\ge i$. Hence the number of sources of μ which $\le b$ is at least $i + 1$, a contradiction. Hence $b \le a$ and so $\mathrm{lft}(\nu) \le \mathrm{rgt}(\mu)$.

Finally, assume that $\mathrm{lft}(\nu) = \mathrm{rgt}(\mu)$. Then in forming the composite $\nu^* \circ \mu$, there are no finite orbits (or contractible loops). For if there were any such orbit, it would be contained in the fundamental rectangle because of the rank condition and hence some element of $u(\{0\} \times \mathbf{n})$ would be in $\mathrm{lft}(\nu) \cap \mathrm{lft}(\mu)$, which is impossible. Hence $\langle \mu, \nu \rangle_{t,0} = 1$. □

As an immediate consequence of (4.4.1) and (4.6), we have

(4.7) COROLLARY. *If* $(t, z) \in \Lambda^a$ *and* $n \in \mathbf{Z}_{\ge 0}$, *we have*

$$\det G_{t,z}(n) = \pm \prod_{\substack{r > t \\ r \equiv t \mod 2}} (z^2 - q^r - q^{-r} + z^{-2})^{\dim W_{r,z}(n)} .$$

(4.8) COROLLARY.

(1) *If n is an odd positive integer, then Jones' annular algebra* $\mathbf{J}(n)$ *(with parameter $\delta = -q - q^{-1}$) is non-semisimple if and only if there exist distinct odd integers $s, t \in \mathbf{n}$ such that $q^{st} = 1$.*

(2) *If n is an even positive integer, then Jones' annular algebra* $\mathbf{J}(n)$ *(with parameter $\delta = -q - q^{-1}$) is non-semisimple if and only if $q^{\frac{n}{2}+1} = 1$ or there exist distinct even integers $s, t \in \mathbf{n}$ such that $q^{\frac{st}{2}} = 1$.*

Proof. By [GL, 3.8] the algebra is semisimple precisely when the bilinear pairing $\langle\ ,\ \rangle_{t,z}$ is non-degenerate on each cell representation (of $\mathbf{J}(n)$); this condition is equivalent to the vanishing of the determinant $\det G_{t,z}(n)$, which by (4.7) immediately yields the stated condition. $\square$

§5. DECOMPOSITION MATRICES

(5.1) THEOREM. *Let R be an algebraically closed field of characteristic zero and q a nonzero element of R. Let $\preceq$ be the weakest partial order on the set Λ^a defined in (2.6) such that $(t,z) \preceq (s,y)$ if (t,z) and (s,y) satisfy the hypotheses of Theorem (3.4) for q or q^{-1}. If $(t,z) \in \Lambda^a$, $n \in \mathbf{Z}_{\geq 0}$ and $(s,y) \in \Lambda^a(n)$, then the multiplicity of the irreducible $\mathbf{T}^a(n)$-module $L_{s,y}(n)$ in the cell representation $W_{t,z}(n)$ of (2.6) is one if $(s,y) \succeq (t,z)$ and zero otherwise.*

Proof. Let R be a field and $q \in R$. Let $p\colon R[\mathrm{y}] \to R$ be the R-algebra homomorphism defined by $\mathrm{y} \mapsto q + q^{-1}$, where y is an indeterminate over R. Suppose W is a free $R[\mathrm{y}]$-module of finite rank with an $R[\mathrm{y}]$-bilinear form $\langle\ ,\ \rangle\colon W \times W \to R[\mathrm{y}]$. If R is regarded as a $R[\mathrm{y}]$-module via the homomorphism p, the free R-module $W_R = R \otimes_{R[\mathrm{y}]} W$ inherits an R-bilinear form $\langle\ ,\ \rangle_R\colon W_R \times W_R \to R$ given by $\langle 1 \otimes x, 1 \otimes y\rangle_R = p(\langle x, y\rangle)$. Choose $R[\mathrm{y}]$-bases B_1 and B_2 of W and let G denote the associated gram matrix of $\langle\ ,\ \rangle$. If this form is nonsingular (i.e. $\det G \neq 0$), then it may be shown that the multiplicity of the polynomial $\mathrm{y} - q - q^{-1}$ in the determinant $\det G$ is greater than or equal to the R-dimension of the radical of $\langle\ ,\ \rangle_R$. In fact if we denote the multiplicity of the polynomial $\mathrm{y} - q - q^{-1}$ in $f \in R[\mathrm{y}]$ by $\mathrm{mult}(f)$, then

$$\mathrm{mult}(\det G) = \sum_{i>0} \dim \mathrm{rad}^i$$

where rad^i denotes the image under $\phi\colon W \to W_R : w \mapsto 1 \otimes w$ of the $R[\mathrm{y}]$-submodule $\{w \in W \mid \langle w, v\rangle \in (\mathrm{y} - q - q^{-1})^i R[\mathrm{y}] \text{ for any } v \in W\}$.

(Since $R[\mathrm{y}]$ is a principal ideal domain, row and column operations may be used to reduce the proof of this fact to the easy case when G is diagonal.) We shall use this elementary result to give a bound for the dimension of the radical of the restriction of $\langle\ ,\ \rangle_{t,z}$ to $W^s_{t,z}(n)$.

Let $t \leq s$ be non-negative integers of the same parity, $n \in \mathbf{Z}_{\geq 0}$ and assume the hypotheses of the statement. Consider $\mathbf{T}^a_{(R[\mathrm{x}],-\mathrm{x})}$. We shall compute the determinant of the gram matrix $G^s_{t,0}(n)$ as a polynomial in $\mathrm{y} = \mathrm{x} + \mathrm{x}^{-1}$. Our first goal is to compute the multiplicity of $\mathrm{y} - q - q^{-1}$ in this polynomial, i.e. to compute $\mathrm{mult}(\det G^s_{t,0}(n))$. Let l denote the order of q^2. Since $[n]_{\mathrm{x}}$ and $\left[\begin{smallmatrix} n \\ i \end{smallmatrix}\right]_{\mathrm{x}}$ are polynomials in $\mathrm{y} = \mathrm{x} + \mathrm{x}^{-1}$ we may speak of the multiplicity of $\mathrm{y} - q - q^{-1}$ in these polynomials and it is straightforward that

$$\mathrm{mult}\,[n]_{\mathrm{x}} = \begin{cases} 1 & \text{if } l \neq 1, \infty \text{ and } l \text{ divides } n, \\ 0 & \text{otherwise,} \end{cases}$$

$$\text{and hence} \quad \mathrm{mult} \begin{bmatrix} n \\ i \end{bmatrix}_{\mathrm{x}} = \begin{cases} 1 & \text{if } l \neq \infty \text{ and } \mathrm{res}_l(n) < \mathrm{res}_l(i), \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathrm{res}_l(n) \in \{0, 1, \dots, l-1\}$ is determined by $\mathrm{res}_l(n) \equiv n \mod l$.

We next give an expression for $\mathrm{mult}([t;r]_{\mathrm{x}}/[s;r]_{\mathrm{x}})$. Let $r \geq s$ have the same parity as s (or t) and write $X = \{0, 1, \dots, l-1\}$. Then there exist unique elements $k \in \mathbf{Z}$ and $\bar{r} \in X$ such that $r = kl + \bar{r}$. Let $\bar{t}$ denote the unique element of X such that $kl + \bar{t} \equiv \pm t \mod 2l$; define $\bar{s}$ similarly. Define:

$$\epsilon^s_t(r) = \begin{cases} 1 & \text{if } \bar{s} \leq \bar{r} < \bar{t}, \\ -1 & \text{if } \bar{t} \leq \bar{r} < \bar{s}, \\ 0 & \text{otherwise.} \end{cases}$$

The function $\epsilon^t_s(r)$ satisfies

(1) $\epsilon^t_s(r) = \epsilon^{-t}_s(r) = \epsilon^{t+2l}_s(r)$

(2) $\epsilon^s_t(r) = -\epsilon^t_s(r)$.

It is easy to see that if $0 \leq t \leq s \leq r$, then

$$\epsilon^s_t(r) = \mathrm{mult}([t;r]_{\mathrm{x}}/[s;r]_{\mathrm{x}}).$$

By Corollary (4.5) and Proposition (4.6), we have

$$\mathrm{mult}(\det G^s_{t,0}(n)) = \sum_{\substack{r \geq s \\ r \equiv t \mod 2}} \epsilon^s_t(r) \dim W_r(n). \tag{5.1.1}$$

If $l = \infty$ or $s \equiv t$ or $-t \mod 2l$, then $\epsilon^s_t(r) = 0$ and so the multiplicity (5.1.1) is zero. For the remainder of this paragraph, assume that $l \neq \infty$ and

$s \not\equiv \pm t \mod 2l$. Let $t' \in \mathbf{Z}$ be minimal such that $t' > s$ and $t' \pm t \equiv 0 \mod 2l$. Let $s' \in \mathbf{Z}$ be maximal such that $t' > s'$ and $s' \pm s \equiv 0 \mod 2l$. Then $s+2l > t' > s' \geq s > t$. Now in order to compute $\operatorname{mult}(\det G^s_{t,0}(n))$, we partition the sum on the right side of (5.1.1) into three parts:

(1) $s \leq r < s'$.

(2) $s' \leq r < t'$.

(3) $t' \leq r$.

For the terms in the first part, $\epsilon^s_t(r) = 0$. For those in the second part $\epsilon^s_t(r) = 1$ and consequently, these terms contribute $\dim W^{t'}_{s',0}(n) = \sum_{s' \leq r < t'} \dim W_r(n)$ to the sum. The terms in the third part have $\epsilon^s_t(r) = -\epsilon^{t'}_{s'}(r)$ (by properties (1) and (2) of the function $\epsilon^s_t(r)$) and so these terms contribute $\operatorname{mult}(\det G^{t'}_{s',0}(n))$ to the sum.

It follows that

$$\text{(5.1.2)} \qquad \operatorname{mult}(\det G^s_{t,0}(n)) = \dim W^{t'}_{s',0}(n) - \operatorname{mult}(\det G^{t'}_{s',0}(n)).$$

Note that equation (5.1.2) should be interpreted as a recurrence relation for $\operatorname{mult}(\det G^s_{t,0}(n))$, which together with the initial condition $\operatorname{mult}(\det G^s_{t,0}(n)) = 0$ if $n \leq t$, determines the multiplicity.

Now fix $n \in \mathbf{Z}_{\geq 0}$. Choose $(t,z) \in \Lambda^a$ such that $t \leq n$ and $t \equiv n \mod 2$. To prove the Theorem, we shall construct a composition series for $W_{t,z}(n)$.

If (t,z) is maximal in $\Lambda^a(n)$ (with respect to $\prec$), then it follows from Corollary 4.4 and Proposition 4.6, that $\operatorname{rad}_{t,z}(n) = 0$; hence the irreducible module $L_{t,z}(n)$ coincides with $W_{t,z}(n)$ and the statement follows.

Assume that (t,z) is not a maximal element of $\Lambda^a(n)$ and choose $(s,y) \in \Lambda^a(n)$ such that $(s,y) \succ (t,z)$ and s is minimal with respect to this property. Then the hypotheses of Theorem (3.4) are satisfied (possibly after replacing q by q^{-1}) and so we have an injective homomorphism $\theta_n \colon W_{s,y}(n) \to W_{t,z}(n)$ of $\mathbf{T}^a_{R,q}(n)$-modules. The quotient $Q = W_{t,z}(n)/\operatorname{Im}\theta_n$ has basis $\mu + \operatorname{Im}\theta_n$ indexed by standard diagrams $\mu \colon t \to n$ of rank strictly less than $(s-t)/2$. By (2.8), the image of θ_n is contained in $\operatorname{rad}_{t,z}(n)$, whence the bilinear form $\langle\ ,\ \rangle_{t,z}$ descends to $Q \times Q \to R$; its gram matrix (with respect to the basis above) is $G^s_{t,z}(n)$ and $L_{t,z}(n)$ is the quotient of Q by its radical which we denote by $\operatorname{rad}^s_{t,z}(n)$. Consider, for the moment, $\mathbf{T}^a_{R[x],x}$. The multiplicity $\operatorname{mult}(\det G^s_{t,z}(n)) = \operatorname{mult}(\det G^s_{t,0}(n))$ by Corollary (4.4); it follows from the remarks concerning linear algebra at the beginning of this proof that

$$\text{(5.1.3)} \qquad \dim \operatorname{rad}^s_{t,z}(n) \leq \operatorname{mult}(\det G^s_{t,0}(n)).$$

If the order l (of q^2) is infinite, then (s,y) is the unique element of Λ^a such that $(s,y) \succ (t,z)$. If l is finite and $s \equiv t$ or $-t \mod 2l$, then (s,y) is the unique element of Λ^a which covers (t,z). In either case, we saw above that $\operatorname{mult}(\det G^s_{t,0}(n)) = 0$ and so $\operatorname{rad}^s_{t,z}(n) = 0$. Therefore $Q = L_{t,z}(n)$ and the composition factors of $W_{t,z}(n)$ are $L_{t,z}(n)$ together with those of $W_{s,y}(n)$, as required.

Assume that l is finite and $s \not\equiv \pm t \mod 2l$. Let s' and t' be as above and $y' = \epsilon y^{-1}$ where $\epsilon = q^{(s+s')/2} = \pm 1$. Then (s',y') is the unique element of Λ^a such that $(s',y') \succ (t,z)$ and $(s',y') \not\preceq (s,y)$. If $s' > n$, then the initial condition associated with (5.1.2) shows that $\operatorname{mult}(\det G^s_{t,0}(n)) = 0$ and so $\operatorname{rad}^s_{t,z}(n) = 0$; hence $Q = L_{t,z}(n)$ and the statement of (5.1) follows as in the previous paragraph.

Finally, assume that $s' \leq n$. By Theorem (3.4) (with q^{-1} replacing q), there exists an injective $\mathbf{T}^a(n)$-homomorphism $\theta'_n \colon W_{s',y'}(n) \to W_{t,z}(n)$. Thus $L_{s',y'}(n)$ is a composition factor of $W_{t,z}(n)$. Arguing by induction in the poset Λ^a, we may assume that $L_{s',y'}(n)$ is not a composition factor of $W_{s,y}(n) \cong \operatorname{Im}(\theta_n)$ since $(s',y') \not\preceq (s,y)$. It follows that the irreducible module $L_{s',y'}(n)$ is a composition factor of $\operatorname{rad}^s_{t,z}(n)$ and we have, using (5.1.3),

$$\dim L_{s',y'}(n) \leq \dim \operatorname{rad}^s_{t,z}(n) \leq \operatorname{mult}(\det G^s_{t,0}(n)).$$

Arguing as above with (s',y') in place of (t,z) we have

$$\dim L_{s',y'}(n) = \dim Q' - \dim(\operatorname{rad}^{t'}_{s',y'}(n)) \geq \dim W^{t'}_{s',y'}(n) - \operatorname{mult}(\det G^{s'}_{t',0}(n)).$$

Now (5.1.2) asserts that the two ends of this chain of inequalities are equal. Hence we have equality at every step and in particular $L_{s',y'}(n)$ is isomorphic to $\operatorname{rad}^s_{t,z}(n)$. Thus the composition factors of $W_{t,z}(n)$ are $L_{t,z}(n)$ (if $q^2 \neq 0$ or $(t,z) \neq (0,q)$) and $L_{s',y'}(n)$ together with those of $W_{s,y}(n)$, as required. □

(5.2) COROLLARY. *Assume the hypotheses and notation of Theorem 5.1 and let* $\mathbf{J}(n)$ *be Jones' annular algebra (see (2.10)). If* $(t,z) \in \Lambda^a(n)$ *is such that* $t > 0$ *and* $z^t = 1$*, then the* $\mathbf{J}(n)$*-module* $W_{t,z}(n)$ *has composition factors* $L_{s,y}(n)$ *indexed by* $(s,y) \in \Lambda^a(n)$ *such that* $(s,y) \succeq (t,z)$*. The remaining cell module* $W_{0,q}/M$ *(2.10) has composition factors* $L_{s,y}(n)$ *indexed by* $(s,y) \in \Lambda^a(n)$ *such that* $(s,y) \succeq (0,q)$ *and* $(s,y) \not\preceq (2,1)$.

The next result is implicit in [DJ] and may be found in [Ma], [GW] and [W].

(5.3) THEOREM. *Let R be a field of characteristic zero, let q be a nonzero element of R and let $\mathbf{T}(n) = \mathbf{T}_{R,q}(n)$ be the Temperley-Lieb algebra over R, with parameter q. If $n, t \in \mathbf{Z}_{\geq 0}$ and $s \in \Lambda(n)$ (2.3) then the multiplicity of the irreducible $\mathbf{T}(n)$-module $L_s(n)$ in the cell representation $W_t(n)$ (2.2) is one if*

(1) $s = t$, *or*

(2) q^2 *has finite order* l, $t + 2l > s > t$ *and* $s + t + 2 \equiv 0 \mod 2l$,

and zero otherwise.

Proof. Adopt the notation of the proof of (5.1). Let $t \in \Lambda(n)$ and note that $G_t(n) = G_t^{t+2}(n)$. If there is no element $s \in \Lambda(n)$ such that (2) holds, then the computations above show that $\operatorname{mult}(\det G_t(n)) = 0$; hence $W_t(n)$ is irreducible and the statement follows. If q^2 has finite order l and $s \in \Lambda(n)$ satisfies (2), then Corollary (3.5) provides a nonzero homomorphism of $\mathbf{T}(n)$-modules $\theta_n \colon W_s(n) \to W_t(n)$. Hence $L_s(n)$ is a composition factor of $W_t(n)$ and we have

$$\dim L_s(n) \leq \dim \operatorname{rad}_t(n) \leq \operatorname{mult}(\det G_t(n))$$

as in the previous proof. However,

$$\dim L_s(n) = \dim W_s(n) - \dim \operatorname{rad}_s(n) \geq \dim W_s(n) - \operatorname{mult}(\det G_s(n)).$$

Now (5.1.2) again asserts that the ends of this chain of inequalities are equal. Therefore we have equality at each step and in particular $L_s(n)$ is isomorphic to $\operatorname{rad}_t(n)$. $\square$

(5.4) REMARKS.

(1) The decomposition matrices in Theorems (5.1) and (5.3) are "independent of n"; one may therefore speak of the multiplicity of $L_{s,y}$ in $W_{t,z}$ and of L_s in W_t.

(2) Since the dimension of $W_{t,z}(n)$ is known (1.12), the multiplicities of (5.1) may be used to give formulae for the dimensions of the irreducible modules $L_{t,z}(n)$. These formulae are just the inversions of the relations

$$\binom{n}{(n-t)/2} = l_{t,z}(n) + \sum_{\substack{(s,y) \in \Lambda^a \\ (s,y) \succ (t,z)}} l_{s,y}(n)$$

where $l_{s,y}(n) = \dim L_{s,y}(n)$. A similar remark applies to the dimensions of the irreducible modules for the Jones and Temperley-Lieb algebras.

(3) The proofs of (5.1) and (5.3) yield the radical series of the modules concerned; $L_{s,y}(n)$ lies in the k-th layer of $W_{t,z}(n)$ if the length of the interval between (s,y) and (t,z) in Λ^a is k. One might expect the layers of the radical series of the cell modules to coincide with the layers (denoted rad^i above) of some "Jantzen filtration" of the cell representation and its bilinear form (after scaling the indices).

(4) If the characteristic of R times the order l of q^2 exceeds the cardinality of n then Theorems (5.1) and (5.3) remain valid without the restriction that R have characteristic zero.

(5) As indicated in (2.9.1), all of our results may be interpreted as statements about the representation theory of TL_n^a; in particular, they illuminate a part of the modular representation theory of the affine Hecke algebra $H_n^a(q)$. One could ask which irreducible representations of the affine Hecke algebra correspond in the Kazhdan-Lusztig parametrization [KL2] to our $L_{t,z}$. A similar comment applies to the connection with the work [Gj].

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