§1. Involutions, diagrams and categories

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(1) *$\mathcal{O}$ contains no points in* $\operatorname{fix}(\phi_1) \cup \operatorname{fix}(\phi_2)$ *; in this case we call $\mathcal{O}$ a* loop.

(2) *$\mathcal{O}$ contains exactly two points in* $\operatorname{fix}(\phi_1) \cup \operatorname{fix}(\phi_2)$ *; in this case we call $\mathcal{O}$ an* arc *and refer to the two fixed points as the* ends *of the arc.*

When the ends of an arc (case (2) above) are not in the same set $\operatorname{fix}(\phi_i)$ ($i = 1$ or 2) we say the orbit is a *through arc*.

Proof. Suppose that an orbit $\mathcal{O}$ contains a point x of $\operatorname{fix}(\phi_1)$ (say). Following [GL, (4.5)], write $(\phi_1\phi_2)_i = \dots\phi_2\phi_1\phi_2$ (i factors) and write $x_i = (\phi_1\phi_2)_i x$ (for $i = 0, 1, 2, \dots$), so that $x_0 = x$ etc. Then clearly $\mathcal{O} = \{x_0, x_1, \dots\}$. If the orbit $\mathcal{O}$ is finite, the result is immediate by the argument in [GL, loc. cit.]. If $\mathcal{O}$ is infinite, then two of its elements lie in the same V-orbit by finiteness, whence there are indices $i < j$ and $k \in \mathbf{Z}$ such that $V^k x_i = x_j$. Acting by ϕ_1 and ϕ_2, it follows that $V^k x_0 = x_{j \pm i} = x_r$ for some $r > 0$. Hence x_r is fixed by ϕ_1. It follows, using the same argument as in [GL, loc. cit.] that $\mathcal{O} = \{x_0, \dots, x_r\}$, which contradicts the infinite nature of $\mathcal{O}$. $\square$

Notice that the proof of (0.8) shows that any infinite H-orbits must be loops. Also, if X is finite, V may (and generally will) be trivial.

§1. Involutions, diagrams and categories

We shall consider various categories in this work whose objects are the non-negative integers $\mathbf{Z}_{\geq 0}$. The morphisms in these categories are defined in terms of "diagrams" and their "composition", whose definition in turn depends on the notion of a "planar involution" (cf. [GL, §6]). In this section we develop a calculus of involutions and diagrams; our principal purpose is the definition of the category $\mathbf{D}^a$ of affine diagrams. These generalise the familiar diagrams which may be used to define the ordinary Temperley-Lieb algebra $\mathbf{T}(n)$.

(1.1) Definition.

(1) A *planar involution* of the totally ordered set P is a permutation ϕ of P such that ϕ^2 is the identity, ϕ has no fixed points and if $x, y \in P$ then $x \leq y \leq \phi(x) \Rightarrow x \leq \phi(y) \leq \phi(x)$.

(2) If t and n are non-negative integers, a *finite diagram* $\alpha\colon t \to n$ is a planar involution ϕ_α of $\mathbf{t\#n}$, where the latter set is defined in (0.6).

If we visualize $\mathbf{t\#n}$ as two horizontal lines in the plane as indicated in (0.6), such a diagram may be represented by a graph with vertex set $\mathbf{t\#n}$ and edges $(x, \phi(x))$ $(x \in \mathbf{t\#n})$. The planar condition then ensures that this graph can be drawn without intersections in the convex hull of $\mathbf{t\#n}$.

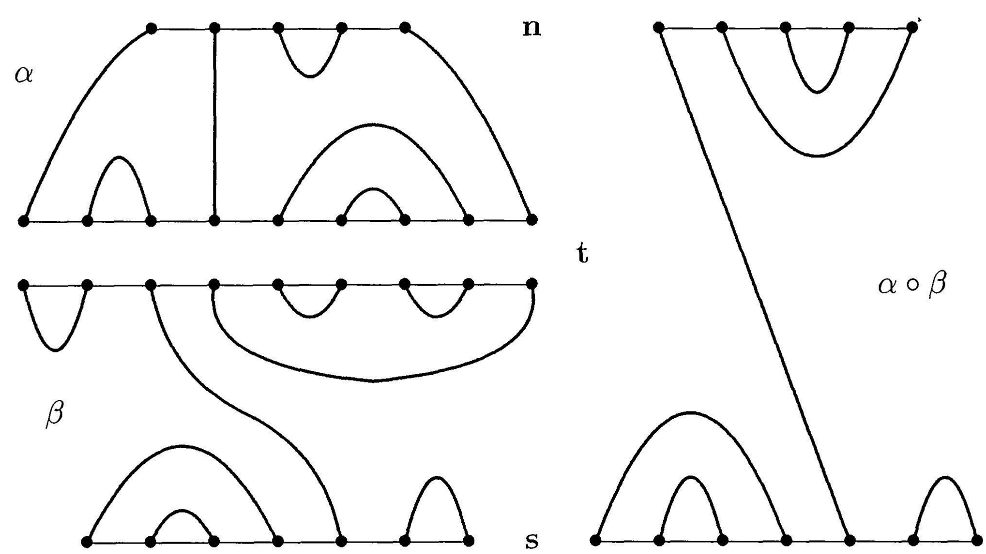
Suppose $\alpha\colon t \longrightarrow n$ and $\beta\colon s \longrightarrow t$ are two finite diagrams with corresponding planar involutions ϕ_α and ϕ_β of $\mathbf{t\#n}$ and $\mathbf{s\#t}$ respectively. We identify $\mathbf{s} \amalg \mathbf{n} = \ell(\mathbf{s}) \cup u(\mathbf{n})$ with its image in $\mathbf{s} \amalg \mathbf{t} \amalg \mathbf{n}$ using the canonical injection. Let $\widetilde{\phi}_\alpha$ denote the involutory bijection of $X = \mathbf{s} \amalg \mathbf{t} \amalg \mathbf{n}$ which fixes $\ell(\mathbf{s})$ and agrees with ϕ_α in the sense that $\widetilde{\phi}_\alpha \circ i_{23} = i_{23} \circ \phi_\alpha$ where $i_{23}\colon \mathbf{t} \amalg \mathbf{n} \to \mathbf{s} \amalg \mathbf{t} \amalg \mathbf{n}$ denotes the canonical injection. Similarly we obtain the involution $\widetilde{\phi}_\beta$ whose fixed point set is $u(\mathbf{n})$. This sets up the situation of (0.8) with $V = \mathrm{id}$.

(1.2) DEFINITION.

(1) With the above notation, let $\phi_{\alpha\circ\beta}$ be the involution of $\mathbf{s\#n}$ which interchanges the ends of the arcs (0.8) of $H = \langle \widetilde{\phi}_\alpha, \widetilde{\phi}_\beta \rangle$ on $\mathbf{s} \amalg \mathbf{t} \amalg \mathbf{n}$. This is a planar involution. Define the *composition* $\alpha \circ \beta$ of α and β to be the diagram corresponding to this involution.

(2) Maintaining the notation of (1), denote by $m(\alpha, \beta)$ the number of loops of H on $\mathbf{s} \amalg \mathbf{t} \amalg \mathbf{n}$. Then $m(\alpha, \beta) = x - (s + n)/2$ where x is the total number of orbits.

In terms of the graphical representation of the diagrams, composition corresponds to placing a graph for α above a graph for β, identifying corresponding points indexed by vertices in $\mathbf{t}$ and deleting the $m(\alpha, \beta)$ interior loops formed. We give an example below.



We define the *category* **D** *of finite diagrams* as follows. Its objects are the non-negative integers. If $t, n \in \mathbf{Z}_{\geq 0}$, the morphisms from t to n are the finite diagrams $\alpha: t \to n$ and composition is as defined in (1.2). The *identity* $\mathrm{id}: t \to t$ interchanges $u(i)$ and $\ell(i)$ for $i \in \mathbf{t}$, in the notation of (0.6).

Next we extend the concept of diagram to the affine case. Let n be a non-negative integer. Recall from (0.7) that $\mathbf{Z} \times \mathbf{n}$ is ordered lexicographically and has an automorphism V_n. The orbits of V_n are represented by the elements of the subset $\{0\} \times \mathbf{n}$.

(1.3) DEFINITION. Let t and n be non-negative integers. An *affine diagram* $\alpha: t \to n$ is a pair $(g(\alpha), \phi_\alpha)$ where $g(\alpha)$ is a non-negative integer and ϕ_α is a planar involution of $(\mathbf{Z} \times \mathbf{t})\#(\mathbf{Z} \times \mathbf{n})$ which commutes with the shift $V_t \# V_n$ (see (0.7)) and which is such that when $g(\alpha)$ is nonzero, ϕ_α preserves the subsets $\ell(\mathbf{Z} \times \mathbf{t})$ and $u(\mathbf{Z} \times \mathbf{n})$.

An affine diagram $\alpha: t \to n$ may be thought of as a graph drawn without intersections on the surface of a cylinder. The lower and upper boundaries of the cylinder have vertices which are labelled by $\ell(\{0\} \times \mathbf{t})$ and $u(\{0\} \times \mathbf{n})$ respectively. Each vertex is joined to another one, the joining curve wrapping around the cylinder a certain number of times, this number being determined by ϕ_α; $g(\alpha)$ denotes the number of closed curves which wrap around the cylinder. The condition that there be no intersections means that if there are any such curves, no top vertex is joined to a bottom vertex (cf. the definition above). In practice it is more convenient to lift such graphs to the universal covering strip of the cylinder, which is the origin of the definition (1.3). We now explain this in detail.

Draw a rectangle (the "fundamental rectangle") in the real plane and extend the horizontal sides indefinitely. Label t points on the lower boundary (avoiding corners) of the rectangle in the obvious way by $\ell(\{0\} \times \mathbf{t})$ and n points on the upper boundary by $u(\{0\} \times \mathbf{n})$. Then label the translates of these points in the translates of the fundamental rectangle to the right and left by $\ell(\mathbf{Z} \times \mathbf{t})$ and $u(\mathbf{Z} \times \mathbf{n})$ in the obvious way. The resulting strip provides a graphical model for $(\mathbf{Z} \times \mathbf{t})\#(\mathbf{Z} \times \mathbf{n})$ (see (0.7)). It is covered by translates of the fundamental rectangle and the shift $V = V_t \# V_n$ moves the fundamental rectangle one step to the right. An affine diagram is depicted in this context by an "augmented graph", drawn without intersections in the strip. This consists of curves joining distinct vertices which are interchanged by ϕ_α, as well as $g(\alpha)$ horizontal curves which stretch along the whole strip, the latter representing closed curves on the cylinder. This graph must be fixed by the translation V. In practice, we draw only the part of the graph in the fundamental rectangle, which determines it completely due to the invariance under V. For the sake of simplicity, the lower and upper vertices of the fundamental rectangle will be labelled in our figures by $\mathbf{t}$ and $\mathbf{n}$ respectively, rather than by the more formal $\ell(\{0\} \times \mathbf{t})$ and $u(\{0\} \times \mathbf{n})$. We shall also use this notation in the text when there is no danger of confusion.

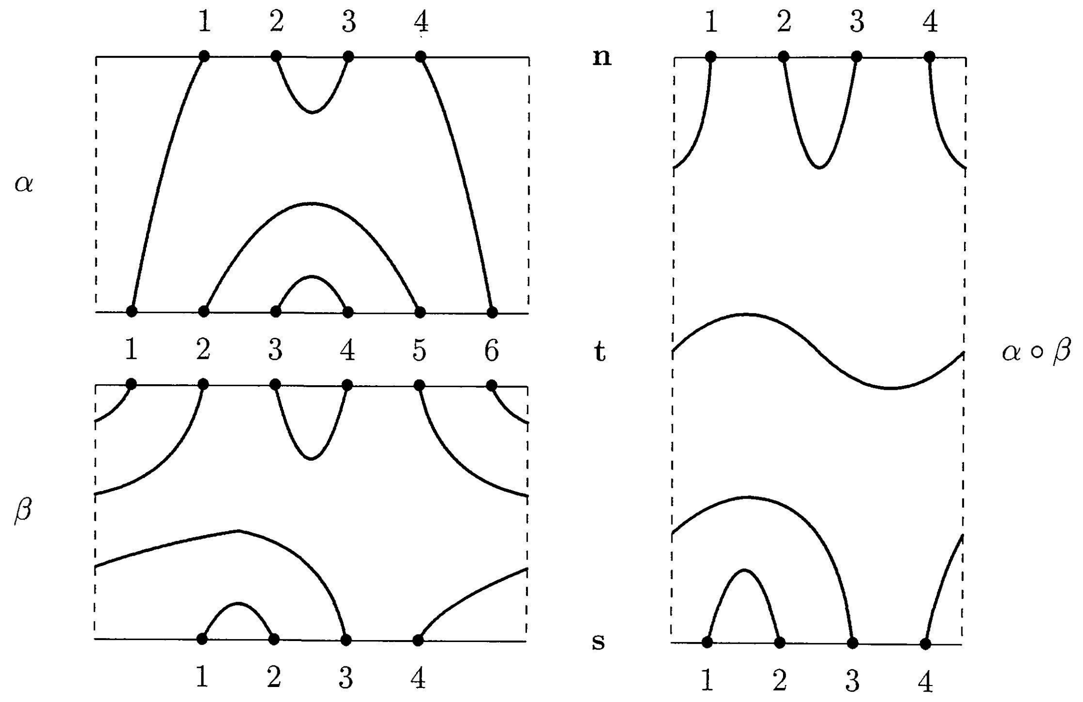
The *rank* $|\alpha|$ of an affine diagram α is the sum of $g(\alpha)$ and the number of vertices $\ell(i,x)$ or $u(i,x)$ with $i < 0$ which are interchanged with vertices $\ell(j,y)$ or $u(j,y)$ with $j \geq 0$. *This is the minimum number of intersections between a graph for α and the left side of the fundamental rectangle.*

If an affine diagram α has rank zero, the restriction of the involution ϕ_α to the fundamental rectangle of α is a finite diagram which characterises α; conversely any finite diagram defines a unique affine diagram (by translating its graph). Thus the finite diagrams from t to n may be thought of as special cases of affine diagrams.

Our earlier definition (1.2) of composition for finite diagrams extends to affine diagrams as follows. Let $\alpha: t \to n$ and $\beta: s \to t$ be affine diagrams. As in the preamble to (1.2), we identify $(\mathbf{Z}\times\mathbf{s})\amalg(\mathbf{Z}\times\mathbf{n}) = \ell(\mathbf{Z}\times\mathbf{s})\cup u(\mathbf{Z}\times\mathbf{n})$ with its image in the disjoint union $(\mathbf{Z}\times\mathbf{s})\amalg(\mathbf{Z}\times\mathbf{t})\amalg(\mathbf{Z}\times\mathbf{n})$, and we extend ϕ_α (resp. ϕ_β) to an involutory bijection $\widetilde{\phi}_\alpha$ (resp. $\widetilde{\phi}_\beta$) of $X = (\mathbf{Z}\times\mathbf{s})\amalg(\mathbf{Z}\times\mathbf{t})\amalg(\mathbf{Z}\times\mathbf{n})$ with fixed point set $\ell(\mathbf{Z}\times\mathbf{s})$ (resp. $u(\mathbf{Z}\times\mathbf{n})$). Denote by H the group of permutations of X which is generated by $\widetilde{\phi}_\alpha$ and $\widetilde{\phi}_\beta$. There is an obvious permutation V of $(\mathbf{Z}\times\mathbf{s})\amalg(\mathbf{Z}\times\mathbf{t})\amalg(\mathbf{Z}\times\mathbf{n})$ whose restrictions to $(\mathbf{Z}\times\mathbf{s})$, $(\mathbf{Z}\times\mathbf{t})$ and $(\mathbf{Z}\times\mathbf{n})$ coincide with the shifts V_s, V_t and V_n of (0.7). This permutation commutes with $\widetilde{\phi}_\alpha$ and $\widetilde{\phi}_\beta$. Therefore V permutes the orbits of H, and since V has finitely many orbits on X, it has finitely many orbits on the set of H-orbits. Let x be this number and let y be the number of H-orbits which are fixed by V. Note that these must be infinite and therefore will correspond to the "horizontal curves" above. By (0.8), we have two types of H-orbits on X. Moreover the loops fall into two types, viz. finite and infinite. These correspond on the cylinder to contractible and incontractible circuits respectively.

(1.4) DEFINITION. Let $\alpha: t \to n$ and $\beta: s \to t$ be affine diagrams and maintain the above notation. Let $m(\alpha,\beta) := x - y - (s+n)/2$ where x and y are defined in the preamble above. The *composition* $\alpha \circ \beta$ of α and β is the affine diagram $(g(\alpha) + g(\beta) + y,\ \phi_{\alpha\circ\beta})$, where $\phi_{\alpha\circ\beta}$ is the planar involution of $(\mathbf{Z}\times\mathbf{s})\#(\mathbf{Z}\times\mathbf{n})$ which interchanges the ends of arcs (0.8) of H on $(\mathbf{Z}\times\mathbf{s})\amalg(\mathbf{Z}\times\mathbf{t})\amalg(\mathbf{Z}\times\mathbf{n})$ (see above).

A graph for the composition may be obtained in the same way as before by placing a strip with a graph for α above one for β, identifying corresponding points labelled by $\mathbf{Z} \times \mathbf{t}$ and deleting the $m(\alpha, \beta)$ finite (V-orbits of) loops formed. In terms of the corresponding graphs drawn on a cylinder, note that only contractible loops are removed. Interior loops which wrap around the cylinder remain; they correspond to infinite loops in the strip. Here is an illustration.



If $\alpha \colon t \to n$ is a diagram, $\alpha = (g(\alpha), \phi_\alpha)$, its *adjoint* $\alpha^* \colon n \to t$ is given by $\alpha^* = (g(\alpha), \phi_{\alpha^*})$, where ϕ_{α^*} is the planar involution of $(\mathbf{Z} \times \mathbf{n})\#(\mathbf{Z} \times \mathbf{t})$ which interchanges elements $\ell(i,j)$ or $u(p,q)$ with $\ell(i',j')$ or $u(p',q')$ precisely when ϕ_α interchanges $u(i,j)$ or $\ell(p,q)$ with $u(i',j')$ or $\ell(p',q')$. Geometrically, this corresponds to reflecting a graph for α in a horizontal line.

The proof of the following lemma is easy and left to the reader.

(1.5) LEMMA. *Let $\alpha \colon t \to n$, $\beta \colon s \to t$ and $\gamma \colon r \to s$ be (affine) diagrams.*

(1) *The composition $\alpha \circ \beta$ is a diagram $: s \to n$.*

(2) *Composition is associative; i.e. we have $(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma)$ and $m(\alpha, \beta) + m(\alpha \circ \beta, \gamma) = m(\beta, \gamma) + m(\alpha, \beta \circ \gamma)$.*

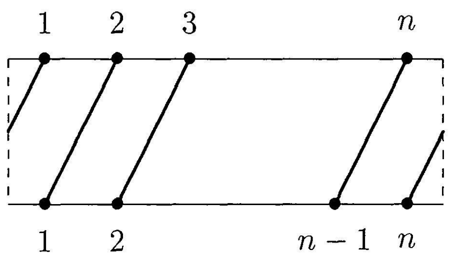
(3) *The finite diagram $id \colon t \to t$ is the identity: $\alpha \circ id = \alpha$ and $id \circ \beta = \beta$.*

(4) *The rank function satisfies $|\alpha \circ \beta| \leq |\alpha| + |\beta|$. Both sides of this inequality have the same parity.*

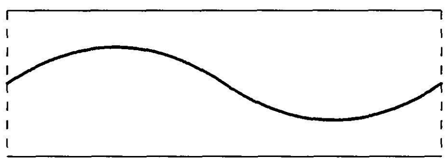
(5) *With the above definition of adjoint, we have $(\alpha \circ \beta)^* = \beta^* \circ \alpha^*$.*

In view of (1.5), we may define the *category* $\mathbf{D}^a$ *of affine diagrams*. This has as objects the non-negative integers and the morphisms from n to m ($n, m \in \mathbf{Z}_{\geq 0}$) are the affine diagrams $\alpha\colon n \to m$. We shall refer to diagrams of even (resp. odd) rank as "even" (resp. "odd").

We now discuss some key examples which play an important rôle in the development below. If σ is any order preserving permutation of $\mathbf{Z} \times \mathbf{n}$, there is a diagram $: n \to n$, also denoted by σ, defined as follows: $\sigma = (0, \phi_\sigma)$, where ϕ_σ is the involution which interchanges lower vertex $\ell(x)$ with upper vertex $u(\sigma(x))$. For example, take $\sigma = \tau_n$ where τ_n is the permutation of $\mathbf{Z} \times \mathbf{n}$ $(n > 0)$ which takes each element to the next largest one. The corresponding diagram $\tau_n\colon n \to n$ appears below. We shall denote by τ_0 the diagram $(1, \phi_{\tau_0})\colon 0 \to 0$ where ϕ_{τ_0} is the unique permutation of the empty set.

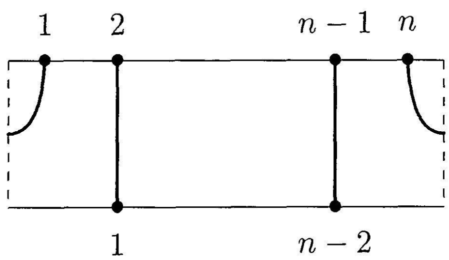


$\tau_n : n \to n$

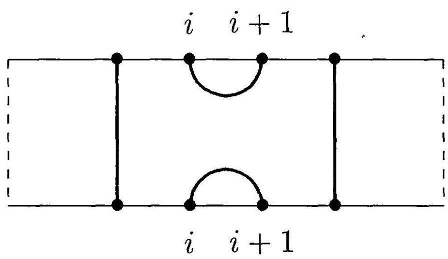


$\tau_0 : 0 \to 0$

Fix an integer $n \geq 2$. Let ϕ_η be the planar involution of $(\mathbf{Z}\times(\mathbf{n}-\mathbf{2}))\#(\mathbf{Z}\times\mathbf{n})$ defined as follows: ϕ_η interchanges the upper vertices $u(0,n)$ and $u(1,1) = V(u(0,1))$ and the vertices $\ell(0,i)$ and $u(0,i+1)$ for $i = 1, 2, \ldots, n-2$. Let $\eta = \eta_n\colon n-2 \to n$ be the affine diagram $(0, \phi_\eta)$. Define $f_0 = \eta \circ \eta^*$ and $f_i = \tau^i \circ f_0 \circ \tau^{-i}$. Note that the f_j are all diagrams $: n \to n$ and that $f_{i+n} = f_i$. Graphs for these diagrams are depicted below.



$\eta_n : n - 2 \to n$



$f_i : n \to n$

We shall usually use τ and η without the subscript, relying on the context to specify it.

Recall that a morphism $f\colon A \to B$ in any category is *monic* if, for any object X and morphisms $i, j\colon X \to A$ we have $f \circ i = f \circ j \Rightarrow i = j$.

(1.6) LEMMA.

(i) *For any diagram* $\alpha\colon t \to n$, *the following are equivalent:*

(1) α *is not monic.*

(2) ϕ_α *interchanges some pair of lower vertices.*

(3) $\alpha = \alpha \circ f_i$ *for some* $f_i\colon t \to t$ *as above.*

(ii) *The monic diagrams* $\sigma\colon n \to n$ *are precisely the powers* τ_n^i *where* $i \in \mathbf{Z}$ *and* $i \geq 0$ *if* $n = 0$.

Proof. (1) $\Rightarrow$ (2): If (2) does not hold, then $\alpha^* \circ \alpha$ is the identity $\mathrm{id}\colon t \to t$ and so α is monic.

(2) $\Rightarrow$ (3): If $x < \phi_\alpha(x)$ are lower vertices as close as possible, then the planar condition ensures that $\phi_\alpha(x)$ covers x. Thus if i is defined by $x = \ell(0, i)$, then $\alpha = \alpha \circ f_i$.

(3) $\Rightarrow$ (1): This is immediate.

Part (ii) follows immediately from (i). □

(1.7) DEFINITION. An (affine) diagram $\mu = (g(\mu), \phi_\mu)\colon t \to n$ is *standard* if μ is monic, $g(\mu) = 0$ and ϕ_μ maps each element of $\ell(\{0\} \times \mathbf{t})$ to $u(\{0\} \times \mathbf{n})$.

The *image* of a diagram $\alpha\colon s \to n$ is the standard diagram constructed as follows. Let $x_1 < x_2 < \cdots < x_t$ be those upper vertices in the fundamental rectangle of $(\mathbf{Z} \times \mathbf{s})\#(\mathbf{Z} \times \mathbf{n})$ which ϕ_α maps to lower vertices and set $t(\alpha) := t$. We refer to $t(\alpha)$ as the number of *through strings* of α. Then the image $i(\alpha)\colon t(\alpha) \to n$ is defined as the monic diagram $i(\alpha) = (0, \phi_{i(\alpha)})$ where $\phi_{i(\alpha)}$ is the involution which interchanges $\ell(0, j)$ with x_j and interchanges upper vertices whenever ϕ_α does. Then any diagram α factors uniquely through its image. Specifically, we have a unique diagram $\rho\colon s \to t(\alpha)$ such that

$$\alpha = i(\alpha) \circ \rho \quad \text{and} \quad \rho^* \text{ is monic.} \tag{1.7.1}$$

If α is monic then ρ is also monic whence $t(\alpha) = s$ and ρ is a power of τ_s.

A particular case of (1.7.1) which we shall use below relates to the case $s = n$. If $\alpha\colon n \to n$ is an affine diagram, there are unique integers $t(\alpha)$, $j(\alpha)$ and standard diagrams $\mu, \nu\colon t(\alpha) \to n$ such that

$$\alpha = \mu \circ \tau_{t(\alpha)}^{j(\alpha)} \circ \nu^* . \tag{1.7.2}$$

(1.8) PROPOSITION. *For any positive integer n, the semigroup generated by the diagrams $f_i : n \to n$ is the set of non-monic diagrams $\alpha : n \to n$ of even rank.*

Proof. If $\alpha : n \to n$ is in the semigroup generated by the f_i, we note that α is even by Lemma 1.5(5) and not monic by the previous lemma.

We prove the converse by induction on length $l(\alpha)$ which is defined by $l(\alpha) = \sum_{i=1}^{n} |\tau^i \circ \alpha \circ \tau^{-i}|$. Let $\alpha : n \to n$ be an even and non-monic diagram. Replacing α by $\tau^i \circ \alpha \circ \tau^{-i}$ if necessary, we may assume that $\alpha \circ f_0 = \alpha$, or equivalently that ϕ_α interchanges the lower vertices $\ell(0,1)$ and $\ell(-1,n)$. Since α is even, it follows that $|\alpha| \geq 2$. We shall construct below a diagram $\beta : n \to n$ such that $l(\beta) = l(\alpha) - 2$ and $\alpha = \beta \circ f_0$. Assuming that β is not the identity, it is clear that β is even and not monic. By induction β is in the semigroup, and thus so is α.

We now construct β leaving it to the reader to verify that one does obtain a diagram with the properties above. In this proof only, let us say that a vertex $v(i,x)$ (where $v = \ell$ or $v = u$) is negative (for α) if $i < 0$; $v(i,x)$ is special if it is negative and $\phi_\alpha v(i,x)$ is not negative. For example $\ell(-1,n)$ is special.
CASE 1: If $g(\alpha) > 0$ and $\ell(-1,n)$ is the only special lower vertex, let $g(\beta) = g(\alpha) - 1$ and ϕ_β be the involution which interchanges the lower vertices $\ell(i,1)$ and $\ell(i,n)$ (for all $i \in \mathbf{Z}$), and acts as ϕ_α elsewhere.
CASE 2: Otherwise our hypotheses ensure that there is an even number of special vertices. Let y be the minimal special vertex excluding $\ell(-1,n)$. Then let $g(\beta) = g(\alpha)$ and take ϕ_β to be the involution which interchanges $\ell(i,1)$ with $V^i \circ \phi_\alpha(y)$, $\ell(i,n)$ with $V^{i+1}(y)$ (for all $i \in \mathbf{Z}$) and which elsewhere agrees with ϕ_α. □

(1.9) COROLLARY. *If $t < n$ are non-negative integers of the same parity, then the map $\mu \mapsto \mu \circ \eta$ is a bijection between standard diagrams $: t+2 \to n$ and standard diagrams $: t \to n$ of nonzero rank. Here $\eta = \eta_{t+2} : t \to t+2$ is the special diagram defined before (1.6) above.*

Proof. The map is well defined and injective, so it suffices to show that it is surjective. If $\nu : t \to n$ is a diagram of nonzero rank, then as in the previous proof we may construct $\mu : t+2 \to n$ (this is the β of the proof of (1.8)) such that $\mu \circ f_0 = \nu \circ \eta^*$ (this replaces α above). In particular, if ν is standard, then μ is also standard and $\nu = \mu \circ \eta$. □

The result above will be applied later in the following iterated form.

(1.9.1) COROLLARY. *Let $t < s \leq n$ be non-negative integers of the same parity and define k by $s = t+2k$. Write $\eta^k = \eta_s\eta_{s-2}\dots\eta_{t+4}\eta_{t+2} \colon t \to s$. Then the map $\mu \mapsto \mu \circ \eta^k$ is a bijection between standard diagrams $: s \to n$ and standard diagrams $: t \to n$ of rank $\geq k$. Moreover we have $|\mu \circ \eta^k| = |\mu| + k$.*

The final result of this section provides a method of counting the number of standard diagrams of a given type.

(1.10) DEFINITION. A standard diagram $\mu \colon t \to n$ determines a partition of the set $u(\{0\} \times \mathbf{n})$ of upper vertices into three parts:

$$\begin{aligned} \mathrm{thr}(\mu) &= \{\phi_\mu(x) \mid x \in \ell(\{0\} \times \mathbf{t})\}, \\ \mathrm{rgt}(\mu) &= \{x \in u(\{0\} \times \mathbf{n}) \setminus \mathrm{thr}(\mu) \mid \phi_\mu(x) < x\}, \\ \mathrm{lft}(\mu) &= \{x \in u(\{0\} \times \mathbf{n}) \setminus \mathrm{thr}(\mu) \mid \phi_\mu(x) > x\}. \end{aligned}$$

The names are intended to reflect the facts that any upper vertex either lies on a "through" arc or is the left or right end of an arc between upper vertices. We shall sometimes abuse notation by writing $i \in \mathrm{thr}(\mu)$ if $u(0, i) \in \mathrm{thr}(\mu)$.

(1.11) PROPOSITION. *If n, t and k are non-negative integers such that $n = t + 2k$, then the map* lft *induces a bijection between the set of standard diagrams $\mu \colon t \to n$ and subsets of cardinality k of $u(\{0\} \times \mathbf{n})$.*

Proof. We prove by induction on k that a standard diagram $\mu \colon t \to n$ is determined by the set $\mathrm{lft}(\mu)$. The case $k = 0$ is trivial. Replacing μ by a conjugate $\tau_n^i \circ \mu \circ \tau_t^{-j}$ if necessary, we may assume that $\mathrm{lft}(\mu)$ contains $u(0, n)$ but not $u(0, 1)$. Since ϕ_μ is planar, the inequality $\phi_\mu(u(1,1)) \leq u(0,n) < u(1,1) \leq \phi_\mu(u(0,n))$ implies $\phi_\mu(u(0,n)) = u(1,1)$. Consequently, $\mu = \eta_n \circ \nu$ where $\nu \colon t \to n-2$ is the standard diagram $\eta_n^* \circ \nu$. By induction $\nu \colon t \to n-2$ is determined by the subset $\mathrm{lft}(\nu) = \{u(0, x-1) \mid u(0,x) \in \mathrm{lft}(\mu),\ x \neq n\}$ and thus $\mu = \eta \circ \nu$ is determined by $\mathrm{lft}(\mu)$. The surjectivity of the map lft is proved in analogous fashion. $\square$

(1.12) COROLLARY. *Let $n, t \in \mathbf{Z}_{\geq 0}$ be integers of the same parity. The number of standard diagrams $\alpha \colon t \to n$ is $\binom{n}{(n-t)/2}$.*