§3. HOMOMORPHISMS AND NATURAL TRANSFORMATIONS

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(2.13) THE TRIVIAL REPRESENTATION OF THE FINITE TEMPERLEY-LIEB ALGEBRAS

The cell module $W_s(s)$ is one-dimensional and will be referred to as the *trivial representation* of $\mathbf{T}(s)$. Observe that the diagrams $f_i \in \mathbf{T}(s)$ all act as the zero operator in this representation, whence if e_s is the corresponding idempotent in $\mathbf{T}(s)$ (e_s exists generically, by generic semisimplicity), then $f_i * e_s = 0 = e_s * f_i$ for all i. The idempotent e_s is referred to in the literature (cf. [MV], [We], [Li] and [J3], where e_s was first identified) as the Jones, or augmentation idempotent of $\mathbf{T}(s)$.

(2.14) LEMMA. *Let t, s and k be non-negative integers such that $s = t + 2k$. If $x \in W_{t,z}(s)$ is annihilated by all finite diagrams $\alpha: s \to s$ except id_s, then x is a scalar multiple of $e_s * \eta^k$, where e_s is defined above and η^k is defined in (1.9.1).*

Proof. We may suppose that $k > 0$, since the case $k = 0$ is trivial. The hypothesis implies that Rx is a realization of the trivial representation of $\mathbf{T}(s)$, whence $x \in e_s * W_{t,z}(s)$. We shall therefore be done if we show that

$$e_s * W_{t,z}(s) = Re_s * \eta^k . \tag{2.14.1}$$

Now η^k is characterised among the standard diagrams $: t \to s$ as the unique diagram of maximal rank (k). If $\mu: t \to s$ is standard and $|\mu| < k$, then $\mu = f_i * \nu$, for some standard diagram $\nu: t \to s$ and $i \in \{1, 2, \dots, s-1\}$ because ϕ_μ must interchange two upper vertices in the fundamental rectangle (recall $k > 0$). Hence $e_s * \mu = e_s * f_i * \nu = 0$, proving (2.14.1) and hence the lemma. □

§3. HOMOMORPHISMS AND NATURAL TRANSFORMATIONS

For any integer n, define the Gaussian integer $[n]_{\mathrm{x}}$ in the function field $\mathbf{Q}(\mathrm{x})$ by

$$[n]_{\mathrm{x}} := \frac{\mathrm{x}^n - \mathrm{x}^{-n}}{\mathrm{x} - \mathrm{x}^{-1}} = \mathrm{x}^{n-1} + \mathrm{x}^{n-3} + \cdots + \mathrm{x}^{1-n} .$$

Define the Gaussian x-factorial by

$$[n!]_{\mathrm{x}} = [n]_{\mathrm{x}}[n-1]_{\mathrm{x}} \dots [2]_{\mathrm{x}}[1]_{\mathrm{x}} .$$

For any pair $n \geq k$ of positive integers, the Gaussian binomial coefficient is

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\mathrm{x}} = \frac{[n]_{\mathrm{x}}[n-1]_{\mathrm{x}}\dots[n-k+1]_{\mathrm{x}}}{[k]_{\mathrm{x}}[k-1]_{\mathrm{x}}\dots[1]_{\mathrm{x}}} \quad \text{and} \quad \begin{bmatrix} n \\ 0 \end{bmatrix}_{\mathrm{x}} = 1 .$$

These are Laurent polynomials in x, so that we may speak of $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q$ for any invertible element q of a ring R. If q^2 has finite order $l > 1$, then $[n]_q = 0$ iff l divides n.

(3.1) DEFINITION. A *forest* is a partially ordered set such that if $x \leq y$ and $x \leq z$ then $y \leq z$ or $z \leq y$.

(3.2) EXAMPLES.

(1) The set of orbits of a planar involution ϕ (of a totally ordered set) is a forest $F(\phi)$ with order defined by: $X \leq Y$ if X is contained in the convex hull of Y. For a finite diagram $\alpha\colon t \to n$, the associated forest for ϕ_α is denoted $F(\alpha)$.

(2) For any affine diagram $\mu\colon t \to n$, order the set $\mathrm{lft}(\mu)$ (see (1.10)) by stipulating that $y \preceq x$ if $x \leq y \leq \phi_\mu(x)$ or $x \leq Vy \leq \phi_\mu(x)$. This condition amounts to the requirement that the convex hull of the orbit of x contains some translate of the orbit of y. The resulting poset is a forest which we denote by $P(\mu)$.

The following result is well known.

(3.3) PROPOSITION (Stanley [RS]). *Let P be a forest of cardinality n; for $y \in P$ denote by h_y the number of elements of P which are less than or equal to y. Then the rational function*

$$h_P(\mathrm{x}) := \frac{[n!]_{\mathrm{x}}}{\prod_{y \in P} [h_y]_{\mathrm{x}}}$$

is a Laurent polynomial with integer coefficients.

It is possible to strengthen the proofs of [RS (5.3) and (22.1)] to yield that the coefficients of $h_P(\mathrm{x})$ are actually positive, but we do not require this here.

The next result, one of the main ones in this work, provides the homomorphisms between cell modules which enable us to analyse them. For any affine diagram $\mu\colon t \to s$, we sometimes (e.g. in the statement of (3.4)) identify the set $u(\{0\} \times \mathrm{s})$ with s in the obvious way, thereby identifying the sets $\mathrm{thr}(\mu)$, $\mathrm{rgt}(\mu)$ and $\mathrm{lft}(\mu)$ with subsets of s.

(3.4) THEOREM. *Let R be a ring with an invertible element q. Let t, s and k be non-negative integers such that $t+2k=s$. Let $z \in R$ be such that $z^2 = q^s$ and set $y = zq^{-k}$, so that $y^2 = q^t$. Then there exists a natural transformation $\theta\colon W_{s,y} \to W_{t,z}$ of $\mathbf{T}^a$-modules (2.5, 2.6) whose component at n applied to a monic diagram $\nu\colon s \to n$ is given by:*

(3.4.1)
$$\theta_n(\nu) = \sum_{\substack{\mu\colon t \to s \\ \text{standard}}} q^i z^{k-|\mu|} h_{P(\mu)}(q)\, \nu\mu$$

where $2i = s(|\mu|-k) + t(s+1)/2 - \sum_{u(x,0)\in \mathrm{thr}(\mu)} x$, $|\alpha|$ is the rank (see (1.3) et seq.) of the affine diagram α and $h_{P(\mu)}(\mathrm{x})$ is the polynomial associated to the forest $P(\mu)$ of (3.2)(2) by (3.3).

Proof. We shall assume without loss of generality that R is the function field $\mathbf{Q}(q^{1/2})$ and that $s>t$, the case $s=t$ having been covered in Theorem (2.8).

To define a natural transformation θ from $W_{s,y}$ to $W_{t,z}$, we require, for each $n \in \mathbf{Z}_{\geq 0}$, a homomorphism $\theta_n\colon W_{s,y}(n) \to W_{t,z}(n)$ such that for any pair n, m of non-negative integers and diagram $\alpha\colon n \to m$, the following diagram commutes:

(3.4.2)
$$\begin{array}{ccc} W_{s,y}(n) & \xrightarrow{W_{s,y}(\alpha)} & W_{s,y}(m) \\ {\scriptstyle \theta_n}\downarrow & & \downarrow{\scriptstyle \theta_m} \\ W_{t,z}(n) & \xrightarrow{W_{t,z}(\alpha)} & W_{t,z}(m) \end{array}$$

Now $W_{s,y}(s)$ is a one dimensional R-module with basis $\mathrm{id}_s\colon s \to s$. Write $\theta_s(\mathrm{id}_s) = \mathbf{v}$. Taking $n=m=s$ and $\alpha=\tau_s$ in (3.4.2) we see that

(1)
$$\tau_s * \mathbf{v} = y\mathbf{v}.$$

Moreover if we take $m=s-2$ and $n=s$, then $W_{s,y}(m)=0$ whence $\alpha * \mathbf{v} = 0$ for any $\alpha \in \mathbf{T}^a(s, s-2)$. In particular, taking $\alpha = \eta_s^*$, we obtain

(2)
$$\eta_s^* * \mathbf{v} = 0.$$

It follows that (1) and (2) are necessary conditions for the particular θ of (3.4.1) to define a natural transformation. We shall prove (1) and (2) shortly, but first show that they are sufficient for the proof of the theorem. Suppose $\mathbf{v} \in W_{t,z}(s)$ is such that (1) and (2) hold.

Then $\mathbf{v} = \sum_{\substack{\mu\colon t\to s \\ \text{standard}}} c_\mu(q,z)\mu$ for certain coefficients $c_\mu(q,z) \in R$. Define θ by

(3.4.3)
$$\theta_n(\nu) = \sum_{\substack{\mu\colon t\to s \\ \text{standard}}} c_\mu(q,z)\nu\mu$$

for any monic diagram $\nu\colon s \to n$. Then (1) implies that the formula (3.4.3) for θ_n defines a unique R-linear map $W_{s,y}(n) \to W_{t,z}(n)$ for each n; these maps are clearly $\mathbf{T}^a(n)$-module homomorphisms. In order to prove that this family of maps defines a natural transformation, fix a diagram $\alpha\colon n \to m$ and standard diagram $\nu\colon s \to n$. If $\alpha \circ \nu$ is also monic, then

$$\begin{aligned}\alpha * \theta_n(\nu) = \alpha * (\nu * \mathbf{v}) = (\alpha\nu) * \mathbf{v} &= (-q-q^{-1})^{m(\alpha,\nu)}(\alpha\circ\nu) * \mathbf{v} \\ &= (-q-q^{-1})^{m(\alpha,\nu)}\theta_m(\alpha\circ\nu) = \theta_m(\alpha * \nu).\end{aligned}$$

On the other hand, if $\alpha\circ\nu$ is not monic, then by Lemma 1.6 there is $i \in \mathbf{Z}_{\geq 0}$ such that $\alpha\circ\nu = \alpha\circ\nu\circ f_i = \beta\circ\eta^*\circ\tau^{-i}$ where $\beta = \alpha\circ\nu\circ\tau^i$; hence we have

$$\alpha * \theta_n(\nu) = (-q-q^{-1})^{m(\alpha,\nu)}(\alpha\circ\nu) * \mathbf{v} = (-q-q^{-1})^{m(\alpha,\nu)} y^{-i}\beta * \eta^* * \mathbf{v} = 0$$

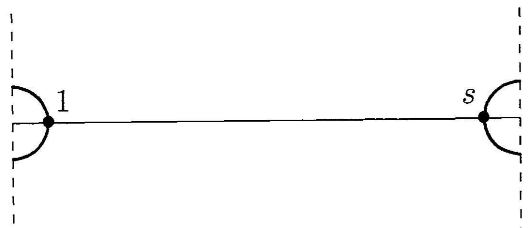
while $\theta_m(\alpha * \nu) = \theta_m(0) = 0$, proving that the squares (3.4.2) commute. It follows that θ is a natural transformation if (1) and (2) hold.

We therefore turn to the proof of (1) and (2) for the particular $\mathbf{v}$ defined by (3.4.1). First we establish (1). Let $\mu\colon t \to s$ be standard, let ν be the image of $\tau\circ\mu$ and recall that $\tau\circ\mu = \nu\circ\sigma$ for some monic diagram $\sigma\colon t \to t$. Suppose first that $s \notin \operatorname{thr}(\mu)$; then σ is the identity. Using the abuse of notation explained after (1.10), we have $\operatorname{thr}(\nu) = \{x+1 \mid x \in \operatorname{thr}(\mu)\}$ and ϕ_ν agrees with $\tau_s\circ\phi_\mu\circ\tau_s^{-1}$ elsewhere. Hence $\sum_{x\in\operatorname{thr}(\nu)} x = t + \sum_{x\in\operatorname{thr}(\mu)} x$, $|\nu| = |\mu| \pm 1$ and $h_{P(\nu)}(\mathrm{x}) = h_{P(\mu)}(\mathrm{x})$. Alternatively, assume that $s \in \operatorname{thr}(\mu)$. Then $t > 0$ and $\sigma = \tau$. We have $\operatorname{thr}(\nu) = \{x+1 \mid x \in \operatorname{thr}(\mu), x \neq s\} \cup \{1\}$ and ϕ_ν agrees with $\tau\circ\phi_\mu\circ\tau^{-1}$ elsewhere. Hence $\sum_{x\in\operatorname{thr}(\nu)} x = t - s + \sum_{x\in\operatorname{thr}(\mu)} x$, $|\nu| = |\mu|$ and $h_{P(\nu)}(\mathrm{x}) = h_{P(\mu)}(\mathrm{x})$. In either case, the coefficient of ν in $y\mathbf{v}$ equals the coefficient of ν in $\tau * \mathbf{v}$ and (1) follows.

To complete the proof of the theorem, it remains only to prove (2). Fix a standard affine diagram $\nu\colon t \to s-2$. We consider standard diagrams $\mu\colon t \to s$ such that ν is the image of $\eta^*\circ\mu$, because these index the terms in the expression (3.4.1) which contribute to the coefficient of ν in $\eta^* * \mathbf{v}$; we shall show that the sum of these contributions is zero. In the figures below, we depict the upper edge of the fundamental rectangle of μ.

Let h denote the coefficient of $\eta\circ\nu$ in $\mathbf{v}$. Let $a' \in \mathbf{s}$ be minimal subject to $a' > 1$ and $\phi_{\eta\circ\nu}(u(0,a')) \notin u(\{0\}\times\mathbf{s})$. Similarly, let $b' \in \mathbf{s}$ be maximal subject to $b' < s$ and $\phi_{\eta\circ\nu}(u(0,b')) \notin u(\{0\} \times \mathbf{s})$. Define $a = a'/2$ and $b = (s+1-b')/2$. We shall consider four types of diagrams μ and compute the contribution of each type to the coefficient of ν in $\eta^* * \mathbf{v}$ separately. Note that the stipulation that ν is the image of $\eta^* \circ \mu$ implies that μ is determined completely by the images $\phi_\mu(u(0,1))$ and $\phi_\mu(u(0,s))$.

CASE 1: $\phi_\mu(u(0,1)) = u(-1,s)$.

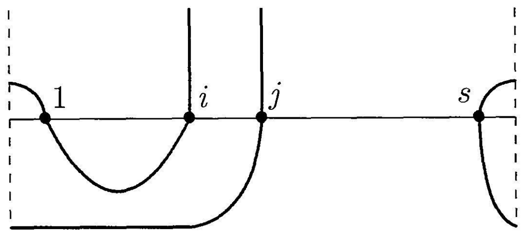


It follows that $\mathrm{thr}(\mu) = \mathrm{thr}(\nu)$ and $\mu = \eta \circ \nu$. Thus $\eta^* * \mu = (-q-q^{-1})\nu$ and so the contribution of the term μ to the coefficient of ν is

$$(-q-q^{-1})h = -[2]_q h\,.$$

CASE 2: Suppose $\phi_\mu(u(-1,s)) > u(0,1)$.

Then $u(0,s) \notin \mathrm{thr}(\mu)$ by planarity. If $\phi_\mu(u(-1,s)) = u(0,j)$ (with $j > 1$), then $\phi_\mu(u(0,1)) = u(0,i)$ and clearly $i < j < a'$ since ϕ_μ is planar.



Since $\eta^* * \mu = \nu$, μ contributes its own coefficient in $\mathbf{v}$ to the coefficient of ν in $\eta^* * \mathbf{v}$. It is easily checked that this coefficient may be expressed as

$$\frac{[1]_q[(j-i+1)/2]_q}{[(j+1)/2]_q[i/2]_q}h\,.$$

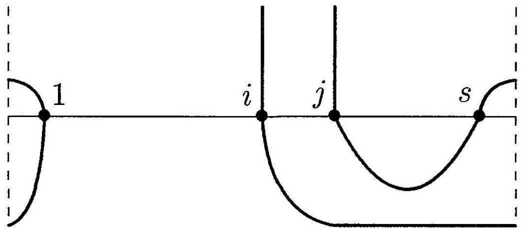
Now the interval $u(0,2), u(0,3), \ldots, u(0,a'-1)$ is a union of $\phi_{\eta\circ\nu}$-orbits. These form a subforest Q of the forest of (3.2)(1) and in this subforest, the $\phi_{\eta\circ\nu}$-orbit $(u(0,i), u(0,j))$ is clearly maximal. Moreover there is an obvious bijection between the maximal orbits $u(0,i_r), u(0,j_r)$ $(r = 1, \ldots, l)$ of $\phi_{\eta\circ\nu}$ on $u(0,2), u(0,3), \ldots, u(0,a'-1)$ and the diagrams μ satisfying the condition $\phi_\mu(u(-1,s)) > u(0,1)$ under which $\phi_\mu(u(-1,s)) = u(0,j_r)$. If (i_1,j_1), (i_2,j_2),..., (i_l,j_l) are the possible pairs (i_r,j_r) as above, listed in order of increasing i_r, then $i_1 = 2$, $j_l = a' - 1$ and $j_k + 1 = i_{k+1}$ for $k = 1, \ldots, l-1$. A straightforward induction argument shows that altogether this family contributes

$$\frac{[a-1]_q}{[a]_q} h$$

to the coefficient of ν.

CASE 3: $\phi_\mu(0,1) < (-1,s)$ and $1 \notin \mathrm{thr}(\mu)$.

If $i = \phi_\mu(1,1)$ and $j = \phi_\mu(s)$, then $b' < i < j$ since ϕ_μ is planar.

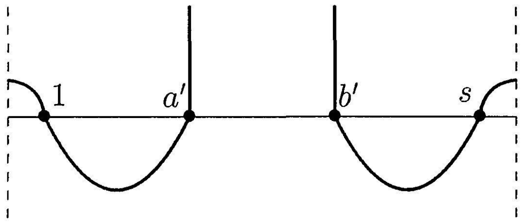


This case is the mirror image of case 2, working from the right instead of the left. Arguing as above, one finds that the total contribution from the μ of this type is

$$\frac{[b-1]_q}{[b]_q} h .$$

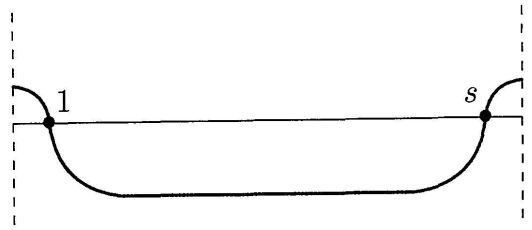
CASE 4: Otherwise.

We shall see that there are just one or two remaining diagrams. First assume that the rank of ν is nonzero. Then it follows from the planar nature of $\eta \circ \nu$ and the choice of a' and b' that $\phi_\eta \circ \nu$ interchanges $u(0,a')$ and $u(i,b')$ for some $i \in \mathbf{Z}$. Since μ is planar, $u(0,1), u(0,s) \notin \mathrm{thr}(\mu)$, so that $\phi_\mu(u(0,1)) > u(0,1)$ and $\phi_\mu(u(0,s)) < u(0,s)$ and it follows that $\phi_\mu(u(0,1)) = u(0,a')$ and $\phi_\mu(u(0,s)) = u(0,b')$.



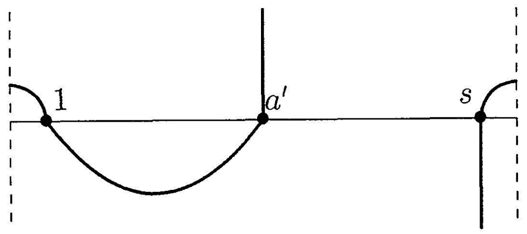
One now computes that this diagram μ contributes $([a+b]_q/([a]_q[b]_q))h$ to the coefficient of ν.

Alternatively, assume that ν has rank zero, i.e. is finite. If $\mathrm{thr}(\nu)$ is empty (in which case $t=0$), then $1, s \notin \mathrm{thr}(\nu)$ (being empty) and it follows that $\mu(u(0,1)) = u(0,s)$.

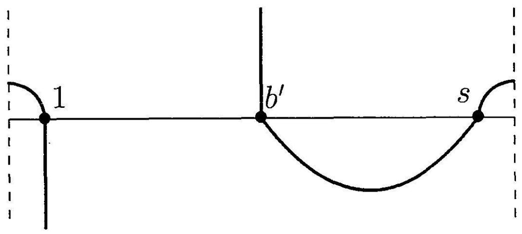


This diagram μ contributes $\chi(\tau_0)q^{-s/2}z^1([1]_q/[s/2]_q)h = ([a+b]_q/([a]_q[b]_q))h$ to the coefficient of ν.

Finally assume that $\mathrm{thr}(\nu)$ is nonempty (and ν is finite). Then $u(0, a'-1)$ and $u(0, b'-1)$ are the minimum and maximum elements of $\mathrm{thr}(\nu)$ respectively. Equivalently, $u(0, a')$ and $u(0, b')$ are the minimum and maximum elements of $\mathrm{thr}(\eta \circ \nu)$ respectively. It follows that either $u(0, s) \in \mathrm{thr}(\mu)$ (in which case $\phi_\mu(u(0,1)) = u(0, a')$)



or $u(0,1) \in \mathrm{thr}(\mu)$ (in which case $\phi_\mu(u(0,s)) = u(0,b')$).



Together these two diagrams contribute

$$\chi(\tau)q^{a-s}z^1\frac{[1]_q}{[a]_q}h + \chi(\tau^{-1})q^{-b}z^1\frac{[1]_q}{[b]_q}h = \frac{[a+b]_q}{[a]_q[b]_q}h\,.$$

We may now compute the sum of the contributions from all four cases:

$$\left(-[2]_q + \frac{[a-1]_q}{[a]_q} + \frac{[b-1]_q}{[b]_q} + \frac{[a+b]_q}{[a]_q[b]_q}\right)h = 0\,.$$

Thus the coefficient of ν in $\eta^* * \mathbf{v}$ vanishes and (2) follows. $\square$

We next prove the following consequence of Theorem (3.4) for the cell modules of the finite Temperley-Lieb algebras. In discussing these, we think of $\mathbf{t}\#\mathbf{n}$ as the fundamental rectangle $\ell(\{0\}\times\mathbf{t}) \cup u(\{0\}\times\mathbf{n})$ of $(\mathbf{Z}\times\mathbf{t})\#(\mathbf{Z}\times\mathbf{n})$ as explained in the discussion after (1.3).

(3.5) COROLLARY. *Let R be a field with an invertible element q. Let t, s be non-negative integers of the same parity such that $t \leq s$. Then there is a natural transformation $\theta\colon W_s \to W_{t,\infty}$ of $\mathbf{T}$-modules (see (2.2) and (2.11.1)) whose component at n applied to a finite monic diagram $\nu\colon s \to n$ is given by*

$$\theta_n(\nu) = \sum_{\substack{\mu\colon t \to s \\ \text{standard}}} h^f_\mu(q)\nu\mu \tag{3.5.1}$$

where $h^f_\mu(\mathrm{x})$ is the polynomial associated in (3.3) to the subforest of $F(\phi_\mu)$ formed by the orbits of ϕ_μ which intersect the fundamental rectangle **t#n** *non-trivially.*

Proof. This result may be established by a computation similar to the one above. However we shall deduce it from Theorem (3.4). First we give a different construction for $W_{t,\infty}$. Let t and s be as above and choose $l', k' \in \mathbf{Z}_{\geq 0}$ such that $k' + l' > s$ and $l' = t + k'$. Set $m = l' + s + k'$ and $z = q^{m/2}$. Define an embedding $\bar{\ }\colon \mathbf{T}(s) \to \mathbf{T}^a(m)$ by mapping f_i to $\bar{f}_i := f_{i+l'}$ for $i = 1, 2, \ldots, s-1$. We say that a monic diagram $\mu\colon 0 \to m$ is *distinguished* if $|\mu| = 0$ and the involution ϕ_μ does not interchange two elements of $u(\mathbf{l}')$ or two elements of $u(\{m, m-1, \ldots, m-k'+1\})$. There is a one to one correspondence ψ between distinguished diagrams $\mu\colon 0 \to m$ and standard diagrams $\nu\colon t \to s$; μ corresponds to ν when ϕ_ν interchanges vertices $u(i)$ and $u(j)$ in $u(\mathbf{s})$ iff ϕ_μ interchanges $u(i+l')$ with $u(j+l')$ in $u(\mathbf{m})$. This defines μ completely, since $\mathrm{lft}(\mu)$ contains $u(\mathbf{l}')$, so that $\mathrm{lft}(\mu)$ is determined, whence μ is, by (1.11).

Suppose $\alpha\colon s \to s$ is finite and $\mu\colon 0 \to m$ is standard. Then $\bar{\alpha} \circ \mu$ is distinguished ($\colon 0 \to m$) only if μ is distinguished. Hence the R-submodule M of $W_{0,z}(m)$ spanned by the non-distinguished standard diagrams $\mu\colon 0 \to m$ is invariant under $\mathbf{T}(s)$. The $\mathbf{T}(s)$-module $W_{0,z}(m)/M$ has basis $\mu + M$ indexed by distinguished diagrams $\mu\colon 0 \to m$, which may be identified using the map ψ above with the standard diagrams $\colon t \to s$. This identification may be extended R-linearly to an isomorphism $\psi\colon W_{0,z}(m)/M \to W_{t,\infty}(s)$ of $\mathbf{T}(s)$-modules.

Now Theorem (3.4) provides an explicit natural transformation $\theta\colon W_{m,1} \to W_{0,z}$. The image $\theta_m(\mathrm{id}_m) = \mathbf{v}$ is given by (3.4.1). Let $\mathbf{w} = \theta^f_s(\mathrm{id}_s)$; i.e. $\mathbf{w}$ is the right hand side of (3.5.1) with $\nu = \mathrm{id}_s$. Then it is easily checked that the isomorphism ψ takes $\mathbf{v} + M$ to $\mathbf{w}$. It follows that $\mathbf{w}$ is annihilated by any non-monic finite diagram and consequently, by an argument similar to that which follows (3.4.2), that the family $\{\theta_n\}$ of homomorphisms given by (3.5.1) defines a natural transformation between the functors W_s and $W_{t,\infty}$. $\square$

(3.6) COROLLARY. *In addition to the hypotheses of the previous corollary, assume that* q^2 *has finite order* $l > 1$. *If* $t < s < t + 2l$ *and* $s + t \equiv -2$ mod $2l$, *then there is a natural transformation* $\theta \colon W_s \to W_t$ *of* $\mathbf{T}$*-modules whose component at* n *is*

$$\theta_n^f(\nu) = \sum_{\substack{\mu\colon t \to s \\ \text{monic} \\ \text{finite}}} h_{F(\mu)}(q)\nu\mu \tag{3.6.1}$$

where $h_{F(\mu)}(\mathrm{x})$ *is the polynomial of (3.3) for the forest of (3.2)(1) associated to the planar involution* ϕ_μ.

Proof. Let $\mu\colon t \to s$ be a monic affine diagram and consider the forest A of orbits of ϕ_μ which intersect $\mathbf{t\#s}$ non-trivially, as in (3.5). Let B be the ideal of A generated by those ϕ_μ-orbits which contain a lower vertex, and let $C = A \setminus B$. If $x \in B$ and $y \in C$, then $x \not\geq y$ and $x \not\leq y$. It follows that

$$h_A(\mathrm{x}) = h_B(\mathrm{x})h_C(\mathrm{x})\begin{bmatrix} a \\ c \end{bmatrix}_{\mathrm{x}}$$

where $h_A(\mathrm{x})$, $h_B(\mathrm{x})$ and $h_C(\mathrm{x})$ are the Laurent polynomials associated by Proposition (3.3) to the forests A, B and C of cardinality a, b and $c = a - b$ respectively. Since $c \leq (s-t)/2 < l$, the denominator $[r!]_{\mathrm{x}}$ of the Gaussian binomial coefficient does not vanish when we set x equal to q and we have

$$\begin{bmatrix} a \\ c \end{bmatrix}_q = \frac{[a]_q \ldots [b+1]_q}{[c]_q \ldots [1]_q}.$$

If μ has nonzero rank, then $a = (t+s)/2 + |\mu| \geq (t+s)/2 + 1 > b$. Since $2l$ divides $s + t + 2$, the numerator vanishes and so $h_{F(\mu)}(q) = h_A(q) = 0$.

Thus the image $\theta_s^f(\mathrm{id})$ of (3.5.1) actually lies in the submodule $W_{t,\infty}^{t+2}(s)$ which is canonically isomorphic to $W_t(s)$. Therefore the right side of (3.6.1) (with $n = s$ and $\nu = \mathrm{id}_s$) is annihilated by non-monic diagrams, and so the argument following (3.4.2) shows that (3.6.1) defines a natural transformation. □

The next result gives an explicit closed formula for the "Jones" or "augmentation" idempotent in the singular case, i.e. when q is a root of unity. There are recursive [We] and partial results concerning formulae for this idempotent, but to our knowledge, the closed formula we give below is new (see also [Li]).

(3.7) COROLLARY. *Assume that q^2 has (multiplicative) order l in R. Then the primitive idempotent (sometimes referred to as the Jones or augmentation idempotent) $e \in \mathbf{T}(l-1)$ which is associated with the trivial representation of $\mathbf{T}(l-1)$ is given by*

$$e = \sum_{\alpha} h_{F(\alpha)}(q)\alpha$$

where the sum is over finite diagrams $\alpha\colon l-1 \to l-1$ and $h_{F(\alpha)}(x)$ is the polynomial associated to the forest of orbits of ϕ_α.

Proof. It clearly suffices to prove that for any non-identity finite diagram $\beta\colon l-1 \to l-1$, we have $\beta * e = e * \beta = 0$. Now the finite diagrams $\alpha\colon l-1 \to l-1$ are in canonical bijection with finite diagrams $\alpha'\colon 0 \to 2l-2$; to see this, imagine the line of lower vertices of α rotated clockwise until it is collinear with the line of upper vertices of α, giving a graph for α'. Moreover if α, β are two finite diagrams $: l-1 \to l-1$, it is easily verified that

(3.7.1) $$(\alpha\beta)' = \beta^* \circ \alpha'$$

where $\beta^* \in \mathbf{T}(l-1)$, regarded as a subalgebra of $\mathbf{T}(2l-2)$ in the usual way i.e. as the subalgebra generated by $\{f_1, \ldots, f_{l-2}\} \in \mathbf{T}(2l-2)$. By (3.6), there is a homomorphism $\theta\colon W_{2l-2}(2l-2) \to W_0(2l-2)$ with image the R-span of $e' := \sum_{\substack{\alpha'\colon 0\to 2l-2 \\ \text{finite}}} h_{F(\alpha')}(q)\alpha'$. But under the identification above, $h_{F(\alpha)}(q) = h_{F(\alpha')}(q)$ for any finite diagram $\alpha\colon l-1 \to l-1$. Hence under the identification, e' corresponds to the element e of the statement. But $\mathbf{T}(l-1)$ clearly acts on this image via the trivial representation. By (3.7.1), it follows that $\mathbf{T}(l-1)$ acts on Re via the trivial representation as required. □

(3.8) REMARK. Part of the significance of (3.7) derives from the fact that the element e is known to generate the radical of Jones' trace function [J1] on the Temperley-Lieb algebra $\mathbf{T}(N)$ (for any N), $\mathbf{T}(l-1)$ being regarded as a subalgebra of $\mathbf{T}(N)$ as explained in the proof of (3.7) and therefore yields a presentation of Jones' projection algebra.

More specifically, Jones (op. cit.) showed that there is a unique trace $tr\colon \mathbf{T}(N) \to R$ which satisfies $tr(1) = 1$ and $tr(xf_i) = \delta^{-1}tr(x)$ for any element $x \in \mathbf{T}(i) \subseteq \mathbf{T}(i+1)$. This trace defines an Hermitian (or bilinear) form on $\mathbf{T}(N)$, which is known to be degenerate if and only if $l \leq N+1$, i.e. $N \geq l-1$. When the Jones form is degenerate, the element $e \in TL(l-1)$ generates (as ideal of $\mathbf{T}(N)$) its radical. Jones' projection algebra $A_{N,\beta}$ [J1] is defined as the quotient of $\mathbf{T}(N)$ by this ideal; hence we obtain an presentation for $A_{N,\beta}$ by simply adding the relation $e = 0$ to the usual presentation of the Temperley-Lieb algebra. For a discussion of other contexts for e, see [MV].

We remark also that it follows from (3.6) (cf. also §5 below) and the theory of cellular algebras that $\mathbf{T}(N)$ is non-semisimple if and only if $N \geq l$. Thus the case $N = l - 1$ is distinguished as the unique one where $\mathbf{T}(N)$ is semisimple, but the Jones form is degenerate.

(3.9) REMARK concerning the Jones (annular) algebras. Since the Jones algebra $\mathbf{J}(n)$ (see (2.10) above) is a quotient of the algebra $\mathbf{T}^a(n)$, any $\mathbf{J}(n)$-module lifts to a $\mathbf{T}^a(n)$-module. The $W_{t,z}(n)$ which correspond to $\mathbf{J}(n)$-modules in this way are those where $z^t = 1$ and $t > 0$ (2.10). Now the conditions $z^2 = q^s$ and $y = zq^{-k}$ (where $s = t + 2k$) of Theorem (3.4) imply (if $t > 0$) that $z^t = 1$ if and only if $y^s = 1$. Hence if $z^t = 1$, the modules $W_{t,z}(n)$ and $W_{s,y}(n)$ of (3.4) may be thought of as $\mathbf{J}(n)$-modules and the map θ_n as a homomorphism of $\mathbf{J}(n)$-modules. If $t = 0$, $z = q$ and the order l of q^2 is finite, then Theorem (3.4) provides a homomorphism $W_{s,y} \to W_{0,q}/M : x \mapsto x + M$ where $s = 2l - 2$, $y = q^l (= \pm 1)$ and M is the module defined in (2.9).

§4. DISCRIMINANTS

(4.1) DEFINITION. Throughout this section R denotes the function field $\mathbf{Q}(q)$ and we consider the affine Temperley-Lieb algebras over the ring $R[z, z^{-1}]$ of Laurent polynomials. If $t \leq s$ are non-negative integers of the same parity define

$$[t; s]_{\mathrm{x}} := \begin{bmatrix} s \\ (s-t)/2 \end{bmatrix}_{\mathrm{x}} .$$

The goal of this section is to compute the *discriminant* of the bilinear pairing

$$\langle\ ,\ \rangle_{t,z} \colon W^s_{t,z}(n) \times W^s_{t,z^{-1}}(n) \to R \quad (n \in \mathbf{Z}_{\geq 0}) .$$

This is the determinant of the *gram matrix* $G^s_{t,z}(n)$ with entries $\langle \mu, \nu \rangle_{t,z}$ indexed by pairs of standard monic diagrams $: t \to n$ of rank (strictly) less than $(s-t)/2$. Recall from (2.12) that these diagrams span a $\mathbf{T}(n)$-submodule $W^s_{t,z}(n)$ of $W_{t,z}(n)$ and that these submodules form an increasing filtration of $W_{t,z}(n)$ as s increases. When $n < s$, we write $G_{t,z}(n)$ for this matrix, because it is then independent of s. Similarly define the gram matrix $G^s_{t,0}(n)$ for the pairing $\langle\ ,\ \rangle_{t,0} \colon W^s_{t,0}(n) \times W^s_{t,\infty}(n) \to R$ and let $G_t(n)$ denote the gram matrix of $\langle\ ,\ \rangle_t \colon W_t(n) \times W_t(n) \to R$ with respect to the basis of finite, monic diagrams. We maintain the standard notation $s - t = 2k$.