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A NEW PROOF OF VINCENT'S THEOREM

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ABSTRACT. Vincent's theorem (1836) asserts that, given a real polynomial $f(x)$ without multiple roots, the substitution

$$x \leftarrow c_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \ddots + \cfrac{1}{c_h + \cfrac{1}{x}}}}$$

where the c_i are arbitrary positive integers and h is sufficiently large, transforms $f(x)$ into a polynomial $f_{h+1}(x)$ which has at most one sign variation in the sequence of its coefficients.

This theorem is basic for highly efficient methods (implemented in modern computer algebra systems) to separate the roots of a real polynomial.

In this paper we provide a new simple proof of the theorem, which improves the known estimates of the size of h and can be extended to the case of multiple roots. We also give an historical survey of the subject.

1. INTRODUCTION

The aim of this paper is to give a new and simple proof of Vincent's theorem. The theorem has an interesting history.

It originally appeared as a note, *Sur la résolution des équations numériques*, appended at the end of the sixth edition of Bourdon's *Élémens d'algèbre* [13], without explicit mention of Vincent's authorship. Bourdon, who was Vincent's father-in-law [1]), merely acknowledges his debt to his son-in-law for "plusieurs améliorations de détail et quelques additions" in the *Avertissement* at the beginning of his book.

[1]) Information about Vincent, who was an influential personality in his time, can be found in [21] and [31].

The debt must have been important, because Vincent later published the result under his name alone: first in the *Mémoires de la Société royale de Lille* (1834), and afterwards, with some improvements, in the *Journal de mathématiques pures et appliquées* (1836) (see [36]).

Unfortunately (for Vincent), Sturm's theorem concerning the number of real roots of an algebraic equation in a given interval, which originally appeared without proof in 1829 and was then published in complete form in 1835, was growing in popularity and ended by superseding Vincent's result. And times were not ripe to understand the remarkable algorithmic potentialities of Vincent's theorem in comparison with Sturm's (see [7]).

Liouville introduces the publication of Vincent's note in his *Journal* with the unflattering remark that the note was being published again, with some additions to the version which had previously appeared in the *Mémoires de Lille*, "dans l'intérêt des professeurs" [36, p. 341, note]. After a subsequent careful reading of Vincent's paper, Liouville commented[2]): "We do not see that these results, curious as they may be, can be of use in our current research."

The theorem was forgotten until 1948, when it was published in Uspensky's book [35]. Uspensky was the first to describe an algorithm based on Vincent's theorem to separate the roots of a polynomial. But to avoid useless calculations, he didn't follow Vincent's original approach (through Budan's theorem), as was pointed out by Akritas ([3], [5]), who also corrected an error in Uspensky's theorem.

Uspensky, who probably doubted that Vincent's original argument could be turned into a proof satisfying modern standards, elaborated another ingenious, but unnecessarily complicated, proof. In Section 6 we show that the essence of Uspensky's result can be obtained through a careful consideration of Vincent's proof.

After Uspensky's book, the theorem appeared in Obreschkoff's book [30], but without any particular application.

The first implementation of an algorithm based on Vincent's theorem in terms suitable for modern computer algebra was made by Akritas (see [1]) and by Rosen and Shallit ([32], see also [18]). Since then, the considerable attention devoted to the subject by Akritas ([3], [5], [6], [7], [8], [9]) has given this algorithm its present status of a powerful tool of computer algebra systems.

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[2]) Quoted in [28, p. 521]. Liouville's text is in a notebook (Ms 3617 (7)) at the Institut de France (Bibliothèque) in Paris. Quite obviously Liouville does not refer only to the content of Vincent's theorem, but to the possibility of using Vincent's result for the studies about transcendental numbers he was conducting at that time.

Curiously, all the proofs before that of Chen-Wang [17], in 1987, have not really used the fact that the complex roots of a real polynomial appear in conjugate pairs. Nor have they considered the effect of the maps of the complex plane into itself, which are naturally related to Vincent's theorem. Chen's proof, which also depends on Obreschkoff's generalization of Descartes' rule of signs, only partially exploits the consideration of the fractional linear transformations connected to Vincent's Theorem, and is rather complicated[3]).

Only Bombieri and van der Poorten consider in full clarity [12] the behaviour of the roots of a polynomial under the action of the fractional linear transformations related to the problem. Proposition 3.1 of [12] gives a result strictly related to Vincent's theorem, regarding the possibility of obtaining *reduced polynomials* (see Remark 8) instead of polynomials having a single sign variation, but the proof can easily be adapted to the situation of Vincent's theorem.

Our proof of the theorem was inspired by the geometric treatment in [12], and combines the use of geometrical transformations with another result of Obreschkoff [30, III, § 17] for which, in a particular but relevant case, we provide a new direct proof.

The resulting proof of Vincent's theorem is simple and short (to us), and can easily be extended to the case of multiple roots[4]).

2. PRELIMINARY FACTS

As we shall deal extensively with sign variations, we begin with

DEFINITION 2.1. Given a sequence (finite or infinite) of real numbers $\alpha_0, \alpha_1, \alpha_2, \ldots$, we say that there is a *sign variation* between two terms α_p and α_q if one of the following conditions holds:

1) $q = p+1$ and α_p and α_q have opposite signs;

2) $q > p+1$ and the terms $\alpha_{p+1}, \alpha_{p+2}, \ldots, \alpha_{q-1}$ are all zero while α_p and α_q have opposite signs[5]).

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[3]) Unfortunately, we haven't yet been able to get Wang's paper [38], and all our information depends on Chen's paper [17]. Hence we refer to Chen-Wang's theorem.

[4]) For the convenience of the reader, we have decided to unify the notation and the symbolism of a subject which, in more than a century and a half, has been considered in very different forms. Throughout the paper the sequence of Fibonacci numbers $F_0, F_1, \ldots$ begins with 1 instead of 0. Some minor changes have been introduced in the statement as well as in the proof of many theorems to conform to this convention.

[5]) Cf. [7, p. 338]

Given an arbitrary real polynomial

$$\alpha_0 x^n + \alpha_1 x^{n-1} + \alpha_2 x^{n-2} + \cdots + \alpha_n ,$$

the expression *(sign) variation of the polynomial* will be used as an abbreviation to mean a sign variation in the sequence of its coefficients.

EXAMPLE 2.2. The polynomial $x^7 - 7x^3 + 3x^2 + 5$, whose sequence of coefficients is $\{1,0,0,0,-7,3,0,5\}$, has two variations, while the polynomial $x^5 - 1$ has one variation.

The idea of relating the number of sign variations of a real polynomial to the number of its positive real roots goes back to the beginning of modern algebra. In his *Géométrie* (1637) Descartes boldly writes[6]) (without any trace of a proof): "An equation can have as many true [positive] roots as it contains changes of sign, from $+$ to $-$ or from $-$ to $+$; and as many false [negative] roots as the number of times two $+$ signs or two $-$ signs are found in succession."

This astonishing claim, which many contemporaries hardly believed, and sometimes misinterpreted[7]), was subsequently improved by the statement that *the number of sign variations of a real polynomial simply is an upper bound to the number of positive roots, the difference being an even number.*

A complete proof was given by Gauss only[8]) in 1828!

Descartes' Rule of Signs, as the previous statement is now called, gives precise information about the positive roots of a polynomial only in two cases: when there are no variations at all and therefore the polynomial has no positive real roots, and when there is a single variation; in the latter case the polynomial has precisely one positive real root.

A deep generalization of Descartes' Rule of Signs is given by the following theorem of Budan and Fourier[9]).

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[6]) [34, p. 160]

[7]) See the letter of Carcavi to Descartes ([19], vol. V, p. 374) and Descartes' answer (ibidem, p. 397).

[8]) See [11] and the review by one of the authors in *Mathematical Reviews* 94d:01017.

[9]) The priority of Budan or Fourier has been a matter of historical dispute for a long time. Fourier's point of view is exposed by Navier in the *Avertissement de l'éditeur* of [11]. From a modern point of view the controversy appears rather pointless. The mere content of the theorem given by the two authors is the same, but Fourier emphasizes its benefit to localize the possible real roots, avoiding the unnecessary calculations that a naive use of Lagrange's "équation au carré des différences" implies ([11, p. 28]). Budan, on the other hand, has an amazingly modern understanding of the relevance of reducing the *algorithm* (his own word) to translate a polynomial by $x = x + p$, where p is an integer, to simple additions [15, pp. 11–16].

THEOREM 2.3. *Consider an n-th degree real polynomial $f(x)$ and two real numbers p, q with $p < q$. Then the sequence*

$$f(p), f'(p), f''(p), \ldots, f^{(n)}(p) \tag{2.1}$$

cannot have fewer variations than the sequence

$$f(q), f'(q), f''(q), \ldots, f^{(n)}(q) . \tag{2.2}$$

The number of real roots of the equation $f(x) = 0$ included in the interval (p, q) equals the difference between the number of variations of the two sequences (2.1) *and* (2.2) *decreased, if necessary, by an even number.*

The choice $p = 0$ and $q = \infty$ immediately yields Descartes' result. The previous theorem provides a better understanding of Descartes' Rule: the role of the single sequence of the coefficients of a polynomial, originally used by Descartes, appears as the result of a very particular situation. Indeed Descartes' Rule is stated in terms of the number of variations of the sequence

$$\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n \tag{2.3}$$

of the coefficients of

$$\alpha_0 x^n + \alpha_1 x^{n-1} + \alpha_2 x^{n-2} + \cdots + \alpha_n \tag{2.4}$$

as a consequence of the fact that the search for the positive roots corresponds to the particular choice of the interval $(0, \infty)$.

In fact, for $x = 0$ the Fourier sequence

$$f(x), f'(x), f''(x), \ldots, f^{(n)}(x)$$

reduces to

$$0! \cdot \alpha_n, \; 1! \cdot \alpha_{n-1}, \; 2! \cdot \alpha_{n-2}, \; \ldots, \; n! \cdot \alpha_0 , \tag{2.5}$$

whose terms differ by a positive factor from the terms of the sequence (2.3). The sequences (2.3) and (2.5) clearly have the same number of variations. For $x = \infty$ the Fourier sequence has no variations. Its role disappears and Descartes' rule may be formulated in terms of a single sequence.

In 1829 Sturm announced the following theorem (proved only in 1835), which seemed to establish definitely [10]) the accidental choice of the sequence (2.3) to investigate the number of positive real roots of the polynomial (2.4).

[10]) The fascinating story of Sturm's theorem as well as the impressive number of algebraic researches it originated is described in [33]. For the sake of simplicity, we state the theorem in the case of the fundamental sequence whose first terms are $f(x)$ and $f'(x)$. Actually Sturm formulated the theorem in the more general terms of what was later called a "Sturm sequence" [7, pp. 341–349].

THEOREM 2.4. *Let $f(x)$ be an n-th degree real polynomial without multiple roots and consider the sequence of polynomials defined recursively by*

$$f_0(x) = f(x), \quad f_1(x) = \frac{df(x)}{dx},$$
$$f_k(x) = q_{k+1}(x) f_{k+1}(x) - f_{k+2}(x), \quad \textit{for} \quad k = 2, \ldots, n-2,$$

where $q_{k+1}(x)$ is the quotient of $f_k(x)$ by $f_{k+1}(x)$ and $f_{k+2}(x)$ is the opposite of the remainder polynomial.

Then the number of zeros of $f(x)$ between p and q ($p < q$) equals the number of variations lost by the sequence

$$f_0(x), f_1(x), \ldots, f_n(x)$$

when $x = p$ is replaced by $x = q$.

Sturm's theorem gives such a clear answer to the problem of determining the number of roots in a given interval that its algorithmic complexity was not considered relevant until the appearence of computer algebra. Let us see how it works through an example.

EXAMPLE 2.5. We take an example from [35]. Given the polynomial

$$f(x) = x^3 + 3x^2 - 4x + 1,$$

we want to know the number of its positive roots. Since $f'(x) = f_1(x) = 3x^2 + 6x - 4$, and we have

$$x^3 + 3x^2 - 4x + 1 = \frac{1}{3}(x+1)(3x^2 + 6x - 4) - \frac{7}{3}(2x - 1),$$

we deduce that

$$f_2(x) = \frac{7}{3}(2x - 1).$$

Again

$$f_1(x) = 3x^2 + 6x - 4 = \frac{9}{14}(x + \frac{5}{2}) \cdot \frac{7}{3}(2x - 1) - \frac{1}{4}$$

and so

$$f_3(x) = \frac{1}{4}.$$

Sturm's sequence is given by

$$\{x^3 + 3x^2 - 4x + 1,\ 3x^2 + 6x - 4,\ \frac{7}{3}(2x - 1),\ \frac{1}{4}\}.$$

For $x = 0$ the sequence becomes

$$\left\{1, -4, -\frac{7}{3}, \frac{1}{4}\right\}$$

and it has two variations. The limits as $x \to +\infty$ give the sequence $\{+\infty, +\infty, +\infty, +\infty\}$, which has no variations. We conclude that $f(x)$ has two positive roots. $\square$

REMARK 1. It is quite evident that Sturm's theorem also makes it possible to *isolate* the roots, i.e. to find disjoint intervals each containing a single root. Consider the previous example. If we evaluate Sturm's sequence at $x = 1$ we have

$$\left\{1, 5, \frac{7}{3}, \frac{1}{4}\right\}.$$

Since this sequence has no variations, the number of variations lost in passing from 0 to 1 is two, and it follows that the positive roots are located in $(0, 1)$. Let us evaluate the sequence for $x = \frac{1}{2}$ following an obvious bisection method. We have

$$\left\{-\frac{1}{8}, -\frac{1}{4}, 0, \frac{1}{4}\right\}.$$

It follows that Sturm's sequence loses one variation in passing from 0 to $\frac{1}{2}$ and loses one more variation in passing from $\frac{1}{2}$ to 1. Hence one root is located in $(0, \frac{1}{2})$ and the other in $(\frac{1}{2}, 1)$.

Considering the complete answer given by Sturm's theorem, the number of variations of a polynomial seems to be very weakly connected to the number of its positive roots, and the 'lucky' case given by 0 or 1 variations looks like an accident.

However we shall see that this situation may be considered the general one. Every polynomial has some sort of 'canonical forms' in which it assumes 0 or 1 variations. Moreover, these canonical forms can be obtained through an algorithm considerably less onerous than the one needed to implement Sturm's theorem.

In the sequel Δ denotes the '*least roots distance*' of the polynomial $f(x)$, that is the minimal distance

$$\min_{j<k} |\alpha_j - \alpha_k|$$

between distinct roots α_i of the equation $f(x) = 0$.

3. A NECESSARY PRELIMINARY STEP: LAGRANGE

As Vincent repeatedly states, an important incentive to develop his own procedure for isolating the roots of an algebraic equation was given by Lagrange's *Traité de la résolution des équations numériques* [26], which collects and improves all the results in [23], [24], [25].

We begin by describing Lagrange's method for approximating a real root of an algebraic equation by a continued fraction expansion, in the oversimplified case of an algebraic equation which has a single positive root.

Actually Lagrange does much more than that, and via his famous *équation au carré des différences*, he gives a method which, in principle, amounts to a complete solution of the problem of approximating all the real roots. Nevertheless his solution is highly impractical and was strongly criticized by Fourier[11]).

Let x_0 be the unique positive root of a polynomial $f(x)$ of degree n, and let the simple continued fraction expansion[12]) of x_0 be given by

$$x_0 = [c_0, c_1, c_2, \dots] = c_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \ddots}}$$

where $c_0 \geq 0$ and $c_i > 0$ for $i > 0$. To avoid trivial cases, we suppose that $x_0 \notin \mathbf{Q}$.

Lagrange's method (see also [12]) consists in constructing a sequence of polynomials $\{f_h(x)\}$ defined recursively by

$$f_0(x) = f(x),$$

and, for $h \geq 0$,

$$f_{h+1}(x) = x^n f_h\big(c_h + \frac{1}{x}\big),$$

where c_h is the integer part (≥ 1 for $h \geq 1$) of the unique positive root

$$\alpha_h = \frac{1}{\alpha_{h-1} - c_{h-1}} \qquad (\alpha_0 = x_0)$$

of the polynomial $f_h(x)$.

Denote the convergents of $x_0 = [c_0, c_1, c_2, \dots]$ by $\dfrac{p_o}{q_0}, \dfrac{p_1}{q_1}, \dfrac{p_2}{q_2}, \dots$. Then

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11) [20, p. 28]. We shall consider this quite interesting question in a subsequent paper.

12) In this paper we make extensive use of the more familiar properties of continued fractions. A concise introduction to the subject is given in [12, Section 2].

(setting, as usual, $p_{-1} = 1$, $q_{-1} = 0$, $p_{-2} = 0$, $q_{-2} = 1$)

$$f_{h+1}(x) = (q_{h-1} + q_h x)^n f\Big(\frac{p_{h-1} + p_h x}{q_{h-1} + q_h x}\Big), \tag{3.1}$$

and equality (3.1) shows that [13])

$$x_0 \in \Big(\frac{p_{h-1}}{q_{h-1}}, \frac{p_h}{q_h}\Big).$$

Each of the polynomials f_h has a unique positive root, and it will be proved later on that, for sufficiently large h, they each have a single variation in the sequence of their coefficients.

This apparently surprising result may be considered a particular case of Vincent's theorem which we are going to examine. But let us begin with a result of Lagrange.

A particularly favourable condition occurs when the variation is located between the coefficients of degree 1 and 0. The possibility of obtaining this particular situation was explored in [26, *Note XII*] for a general change of variables of the form

$$x \leftarrow \frac{p + rx}{q + sx}$$

and for a more general location of the roots, paving the way for future developments which led to Vincent's theorem.

The change of variables

$$x \leftarrow \frac{q}{s} x$$

does not affect the number of variations, consequently Lagrange limited himself to consider

$$x \leftarrow \frac{a + bx}{x + 1}. \tag{3.2}$$

THEOREM 3.1 (Lagrange). *Suppose that the real polynomial $f(x)$ of degree n has a single real root x_0 in the positive interval (a,b) [neither a or b being roots], and that no complex root has its real part in the same interval. If a is chosen sufficiently close to x_0, then the polynomial*

$$\phi(x) = (1 + x)^n f\Big(\frac{a + bx}{1 + x}\Big)$$

has a unique variation, located between the coefficients of degree 0 and 1.

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[13]) By (a,b) we denote the interval whose endpoints are a, b, but we do not suppose $a < b$. We also have $p_{i+1} = c_{i+1}p_i + p_{i-1}$ and $q_{i+1} = c_{i+1}q_i + q_{i-1}$.

Proof. Denote by $x_1, x_2, \dots, x_{n-1}$ the other (real or complex) roots of $f(x)$. Consider first a real root x_j. According to (3.2), x_j is transformed into

$$\xi_j = \frac{x_j - a}{b - x_j}, \tag{3.3}$$

which is positive if and only if $x_j \in (a, b)$, that is if and only if $x_j \equiv x_0$.

Hence the factor $x - x_0$ is transformed into the factor $x - \xi_0$, which has a sign variation, while every other linear factor $x - x_j$ $(j \neq 0)$ is transformed into a factor of the form $x + p$, with $p \in \mathbf{R}^+$.

Consider now a complex root $x_k = \rho_k + i\sigma_k$. Under (3.2), x_k is carried into

$$\xi_k = \frac{\rho_k - a + i\sigma_k}{b - \rho_k - i\sigma_k} = \frac{(\rho_k - a)(b - \rho_k) - \sigma_k^2 + i(b - a)\sigma_k}{(b - \rho_k)^2 + \sigma_k^2}. \tag{3.4}$$

By hypothesis $\rho_k \notin (a, b)$, $(\rho_k - a)(b - \rho_k) < 0$, and hence

$$\operatorname{Re} \xi_k = \frac{(\rho_k - a)(b - \rho_k) - \sigma_k^2}{(b - \rho_k)^2 + \sigma_k^2} < 0.$$

Since complex roots appear in conjugate pairs, (3.2) transforms a quadratic factor of $f(x)$ of the form

$$(x - \rho - i\sigma)(x - \rho + i\sigma) = x^2 - 2\rho x + \rho^2 + \sigma^2$$

into a quadratic factor of the form

$$x^2 + 2Rx + R^2 + S^2,$$

where $R > 0$.

Therefore, $\phi(x)$ is of the form

$$K(x - \xi_0)(x + p) \cdot \ldots \cdot (x^2 + 2Rx + R^2 + S^2) \cdot \ldots ;$$

where all the quantities $\xi_0, p, \dots, R, S, \dots$ are strictly positive, and

$$\xi_0 = \frac{x_0 - a}{b - x_0}. \tag{3.5}$$

Obviously the coefficients of the polynomial

$$(x + p) \cdot \ldots \cdot (x^2 + 2Rx + R^2 + S^2) \cdot \ldots$$

are strictly positive as well. Let us write this polynomial as

$$b_0 x^{n-1} + b_1 x^{n-2} + \cdots + b_{n-2} x + b_{n-1},$$

where $b_i > 0$. Hence, up to the constant K,

$$\begin{aligned}\phi(x) &= (x-\xi_0)(b_0x^{n-1}+b_1x^{n-2}+\cdots+b_{n-2}x+b_{n-1})\\ &= b_0x^n+(b_1-\xi_0b_0)x^{n-1}+(b_2-\xi_0b_1)x^{n-2}+\cdots-\xi_0b_{n-1}\,.\end{aligned}$$

If in (3.5) a is so close to x_0 as to verify

$$\xi_0 < \min\left(\frac{b_1}{b_0}, \frac{b_2}{b_1}, \frac{b_3}{b_2}, \ldots\right),$$

that is,

$$b_1-\xi_0b_0>0, \quad b_2-\xi_0b_1>0, \quad b_3-\xi_0b_2>0, \ \ldots$$

then all the coefficients of $\phi(x)$, with the only exception of the constant term, are positive. $\square$

REMARK 2. The hypothesis on the real parts of the complex roots seems to be a bit artificial, like an 'ad hoc' expedient. A simpler hypothesis is that $|b-a|$ be less than the least distance Δ of all the roots, i.e., $|b-a|<\Delta$. The distance between two conjugate roots $\rho \pm i\sigma$ is 2σ, which entails $\Delta < 2\sigma$. The maximum value of the product $(\rho-a)(b-\rho)$, when $a \le \rho \le b$, is $\frac{1}{4}(b-a)^2$. It follows that

$$\tfrac{1}{4}(b-a)^2 < \tfrac{1}{4}\Delta^2 < \sigma^2,$$

and the real part of the transformed roots given by (3.4) is negative.

REMARK 3. The hypotheses Lagrange makes in *Note XII* are very stringent. By expanding the root into a continued fraction we can find a first integer h sufficiently large in order to have $\left|\dfrac{p_h}{q_h}-\dfrac{p_{h-1}}{q_{h-1}}\right| < \Delta$. This ensures that all the real parts of the roots transformed by

$$x \leftarrow \frac{p_{h-1}+p_hx}{q_{h-1}+q_hx}$$

are negative. Carrying on the process, we can find a second larger integer k such that $\left|\dfrac{p_k}{q_k}-\dfrac{p_{k-1}}{q_{k-1}}\right| < \varepsilon$. Choosing a between $\dfrac{p_k}{q_k}$ and $\dfrac{p_{k-1}}{q_{k-1}}$ and b between $\dfrac{p_h}{q_h}$ and $\dfrac{p_{h-1}}{q_{h-1}}$ we can satisfy Lagrange's condition. But isn't the knowledge of h and k equivalent to the possibility of approximating a root as closely as we desire? At first sight, *Note XII* appears pointless.

4. Vincent's proof of his theorem

A great merit of Vincent is to have understood perfectly the real aim of Lagrange. The requirement that a polynomial have a unique variation at a prescribed place is too demanding. We can be satisfied with the weaker requirement that a polynomial have a unique variation. This weakening gives the endpoints of the interval (a,b) a more balanced role. Moreover, in order to carry out a process for isolating the roots of an algebraic equation $f(x)=0$, it is necessary to consider not only the behaviour of the polynomials f_h corresponding to the continued fraction expansions $[c_0, c_1, c_2, \ldots]$ which approximate the roots, but also the other apparently purposeless expansions – and the related polynomials – which appear out of a systematic search for the roots [14]).

All this will be clarified by Example 5.2. To get to the point in question, let us give a precise statement.

THEOREM 4.1. *Consider an arbitrary real polynomial $f(x)$ of degree n, without multiple roots, and let $\gamma = [c_0, c_1, c_2, \ldots]$, where the c_i are arbitrary positive integers for $i \geq 1$ and $c_0 \geq 0$, the k-th convergent being denoted by $\dfrac{p_k}{q_k}$. Define the sequence of variable substitutions*

$$x \leftarrow c_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \ddots + \cfrac{1}{c_h + \cfrac{1}{x}}}} = \frac{p_{h-1} + p_h x}{q_{h-1} + q_h x}, \qquad h = 0, 1, 2, \ldots$$

Then, for h sufficiently large, the polynomial

$$f_{h+1}(x) = (q_{h-1} + q_h x)^n f\Big(\frac{p_{h-1} + p_h x}{q_{h-1} + q_h x}\Big)$$

has at most one variation.

Proof. To simplify the problem, we again follow Lagrange, setting $a_h = \dfrac{p_{h-1}}{q_{h-1}}$, $b_h = \dfrac{p_h}{q_h}$ and making the substitution $x \leftarrow \dfrac{q_{h-1}}{q_h} x$. We are reduced to studying the variations of the polynomial

$$\phi_{h+1}(x) = (1+x)^n f\Big(\frac{a_h + b_h x}{1+x}\Big). \tag{4.1}$$

14) Via the Budan-Fourier theorem, for example.

For simplicity of notation, we hereafter denote a_h and b_h simply by a and b, and ϕ_{h+1} by ϕ.

Denote again by $x_0, x_1, \ldots, x_{n-1}$ the roots of $f(x)$, and by Δ the least distance between pairs of these roots.

The behaviour of real and complex roots is given by formulae (3.3) and (3.4). But Vincent makes a judicious observation: in order that the root ξ_k obtained from x_k via (3.4) have negative real part, it is enough to require that

$$(\rho_k - a)(b - \rho_k) - \sigma_k^2 < 0\,. \tag{4.2}$$

Considering (4.2) in geometrical terms [15]), we see that it is equivalent to asking that the point (ρ_k, σ_k) of the ρ–σ-plane should lie outside the circle whose equation is

$$\rho^2 + \sigma^2 - (a+b)\rho + ab = 0\,;$$

this circle is centered at $\left(\dfrac{a+b}{2}, 0\right)$ and its radius is $\frac{1}{2}|b-a|$.

But

$$\tfrac{1}{2}|b-a| = \frac{1}{2\, q_h\, q_{h-1}}\,,$$

which shows that, as h increases, $\frac{1}{2}|b-a| \longrightarrow 0$. Condition (4.2) is then satisfied for h sufficiently large.

Assuming that h is large enough to satisfy (4.2) and the further inequality

$$|b-a| = \frac{1}{q_h\, q_{h-1}} < \Delta\,,$$

then at most one real root can belong to the interval (a, b).

Hence, for sufficiently large h, the polynomial (4.1) can be written as

$$K(x \pm \xi_0)(x+p)\cdot \ldots \cdot (x^2 + 2Rx + R^2 + S^2)\cdot \ldots\,,$$

where $p, \ldots, R, S$ are positive and we take the minus or plus sign in $(x \pm \xi_0)$ according to whether or not there exists a real root $x_0 \in (a,b)$ and $\xi_0 = \dfrac{x_0 - a}{b - x_0}$.

Let $g(x)$ be the polynomial whose transformed form under (3.2) is

$$G(x) = (x+p)\cdot \ldots \cdot (x^2 + 2Rx + R^2 + S^2)\cdot \ldots\,.$$

At this point Vincent observes that

$$\frac{a+bx}{1+x} = b + \frac{a-b}{1+x} = b + u\,.$$

---

[15]) Vincent actually uses a slightly different argument. He looks at the minimum value of the product $\left(\frac{p_h}{q_h} - \rho\right)\left(\frac{p_{h-1}}{q_{h-1}} - \rho\right)$.

Hence

$$
\begin{aligned}
G(x) &= (1+x)^{n-1} g(b+u) \\
&= (1+x)^{n-1}\left[g(b) + g'(b)u + g''(b)\frac{u^2}{2!} + \cdots + g^{(n-1)}(b)\frac{u^{n-1}}{(n-1)!}\right].
\end{aligned}
\tag{4.3}
$$

Since $u \to 0$ as $h \to \infty$,

$$G(x) \to g(b)(1+x)^{n-1}$$

and the polynomial (4.1) has the limit

$$K^*(x \pm \xi_0)(1+x)^{n-1}. \tag{4.4}$$

For h large enough, the number of variations of (4.1) is equal to the number of variations of (4.4). If we have the plus sign in the factor $x \pm \xi_0$, there are no variations.

Let us consider the case [16]) given by

$$(x - \xi_0)(1+x)^{n-1}.$$

We have

$$(1+x)^{n-1} = \sum_{k=0}^{n-1} a_k x^{n-1-k}, \quad \text{where} \quad a_k = \binom{n-1}{k},$$

and for $k = 1, 2, \ldots,$

$$a_k^2 - a_{k-1}a_{k+1} > 0. \tag{4.5}$$

Now

$$
\begin{aligned}
(x-\xi_0)(1+x)^{n-1} &= (x-\xi_0)\left[a_0 x^{n-1} + a_1 x^{n-2} + a_2 x^{n-3} + \ldots\right] \\
&= x^n + [a_1 - a_0\xi_0]\,x^{n-1} + [a_2 - a_1\xi_0]\,x^{n-2} + \cdots - \xi_0.
\end{aligned}
$$

From (4.5) it is clear that, if for a given k the coefficient of x^{n-k},

$$a_k - a_{k-1}\xi_0,$$

is negative then all the subsequent coefficients are negative. Since the constant term is negative we have exactly one variation. □

---

[16]) Obreschkoff's lemma, quoted in Section 6, immediately gives the result, but we want to follow Vincent's argument.

REMARK 4. We do not wish to deny Vincent's great value and originality, yet we find his proof disappointing. In fact, after a careful examination of the effect of the variable transformation $x \leftarrow \dfrac{a+bx}{1+x}$ to get information about the location of the roots of the polynomial $\phi(x)$, Vincent abruptly neglects what he has obtained and goes on to consider the effect of Taylor's formula applied to $(1+x)^{n-1}g(b+u)$. This approach carries no trace of all his previous work, and it is evident that the results one can obtain about the size of h are not best possible. A century later Uspensky modified the proof, but followed the same path, as we shall see later. Obviously we are not trying to criticize Vincent, but simply to emphasize the lack of consideration of the complex plane structure.

REMARK 5. While continued fractions appear naturally in the search for the roots of an algebraic equation, and are closely linked to the problem of separating the roots (see the following example), it is evident that they merely provide a tool, in the preceding proof, to get two sufficiently close values a, b. The theorem may be formulated entirely in terms of the transformation (3.2).

EXAMPLE 4.2. To see how Vincent's theorem can be used to separate the roots of an equation, we consider once again the polynomial of Example 2.5. The polynomial $x^3 + 3x^2 - 4x + 1$ has two variations, hence the theorem of Budan and Fourier implies that the equation

$$x^3 + 3x^2 - 4x + 1 = 0 \tag{4.6}$$

has either two or zero positive roots. By making the substitution $x \leftarrow 1 + x$, we obtain the polynomial

$$(1+x)^3 + 3(1+x)^2 - 4(1+x) + 1 = x^3 + 6x^2 + 5x + 1\,,$$

which has no variations and consequently has no positive roots. This shows that the equation (4.6) has no roots greater than 1. To consider the possibility of roots in $(0,1)$, we make the substitution $x \leftarrow \frac{1}{1+x}$. We obtain

$$(1+x)^3\left[\frac{1}{(1+x)^3} + 3\frac{1}{(1+x)^2} - 4\frac{1}{1+x} + 1\right] = x^3 - x^2 - 2x + 1\,.$$

This polynomial still has two variations so it must again be subjected to the transformations $x \leftarrow 1 + x$, $x \leftarrow \dfrac{1}{1+x}$. The transformed polynomials are

$$x^3 + 2x^2 - x - 1\,, \qquad x^3 + x^2 - 2x - 1\,.$$

Each of these has only one variation and hence has exactly one positive root. The first polynomial is obtained by the two successive substitutions

$$x \leftarrow \frac{1}{1+x}, \qquad x \leftarrow 1+x,$$

(or directly by $x \leftarrow \frac{1}{2+x}$). It follows that its positive root corresponds to a root of the original equation (4.6) in the interval $(0, \frac{1}{2})$. The second polynomial is obtained by the two identical substitutions

$$x \leftarrow \frac{1}{1+x}, \qquad x \leftarrow \frac{1}{1+x},$$

or

$$x \leftarrow \frac{1}{1+\dfrac{1}{1+x}}.$$

Its positive root corresponds to a root of (4.6) in $(\frac{1}{2}, 1)$.

REMARK 6. Looking at the previous example, we can be satisfied since the equation

$$x^3 + 3x^2 - 4x + 1 = 0$$

has one positive root in each of the intervals $(0, \frac{1}{2})$, $(\frac{1}{2}, 1)$. At this point, to approximate these roots we could use a suitable method such as Newton's or Dandelin's. But we can also proceed by using the same method. We know that the root of (4.6) in the interval $(\frac{1}{2}, 1)$ corresponds, via the substitution

$$x \leftarrow \frac{1}{1+\dfrac{1}{1+x}},$$

to the positive root of $x^3+x^2-2x-1 = 0$. Inserting the substitutions $x \leftarrow 1+x$, $x \leftarrow \dfrac{1}{1+x}$ into

$$x^3 + x^2 - 2x - 1 = 0$$

gives [17])

$$x^3 + 4x^2 + 3x - 1 = 0\,, \quad 1 + 6x + 5x^2 + x^3 = 0\,.$$

Since only the first equation has a variation, the root of the equation (4.6) is to be found by the transformation

$$x \leftarrow \frac{1}{1+\dfrac{1}{1+(1+x)}} = \frac{1}{1+\dfrac{1}{2+x}}.$$

[17]) The second susbstitution is useless, but we make it for the sake of clarity.

We find the new smaller interval $(\frac{2}{3}, 1)$. This remark explains the double role of Vincent's theorem, to isolate or to approximate the roots.

5. Uspensky's proof of Vincent's theorem

Uspensky had the great merit of rediscovering Vincent's theorem and of providing the first modern proof. He also tried to popularize the use of the theorem as a powerful tool to isolate the roots of algebraic equations, but there he was unsuccessful, and it was only at the end of the seventies, mainly by the work of Akritas, that the root separation algorithm acquired its present status.

To clarify the structure of the proof, which at first sight looks rather cumbersome, we extract part of its content as an independent lemma, which is of little interest in itself, but will be used also in the proof of Section 6.

LEMMA 5.1. *If the n positive numbers*

$$R_k = \binom{n-1}{k}(1+\delta_k)\,, \quad k = 0, 1, \ldots, n-1\,,$$

are such that $|\delta_k| < \dfrac{1}{n}$, then the $n-1$ inequalities

$$R_k^2 - R_{k-1}R_{k+1} > 0\,, \quad k = 1, \ldots, n-1 \tag{5.1}$$

hold.

Proof. The inequalities (5.1) may be written as

$$\frac{(1+\delta_k)^2}{(1+\delta_{k-1})(1+\delta_{k+1})} > 1 - \frac{n}{(n-k)(k+1)}\,. \tag{5.2}$$

If $\varepsilon = \max\{|\delta_k|\}$, the left hand side of (5.2) is greater than

$$\frac{(1-\varepsilon)^2}{(1+\varepsilon)^2} = 1 - \frac{4\varepsilon}{(1+\varepsilon)^2}\,.$$

Hence (5.2) holds if

$$\frac{4\varepsilon}{(1+\varepsilon)^2} < \frac{n}{(n-k)(k+1)}\,. \tag{5.3}$$

The minimum value of

$$\frac{n}{(n-k)(k+1)}$$

is

$$\frac{4n}{(n+1)^2} = \frac{4/n}{(1+\frac{1}{n})^2}.$$

It follows that (5.3) holds if $\varepsilon < \dfrac{1}{n}$. $\square$

Now we give a precise statement, followed by a summary of the essential points of the proof [35, pp. 298–303].

THEOREM 5.2. *Let $f(x)$ be a real polynomial of degree n, without multiple roots, and with least roots distance Δ. Let $\gamma = [c_0, c_1, c_2, \dots]$, where the c_i are arbitrary positive integers for $i \geq 1$ and $c_0 \geq 0$, the k-th convergent being denoted by $\dfrac{p_k}{q_k}$. Let F_k denote the k-th term of the Fibonacci sequence (defined by $F_0 = F_1 = 1$, and $F_k = F_{k-1} + F_{k-2}$ for $k > 1$). If the integer h is such that*

$$F_{h-1}\frac{\Delta}{2} > 1 \quad \text{and} \quad \Delta F_h F_{h-1} > 1 + \frac{1}{\varepsilon_n},$$

where

$$\varepsilon_n = \left(1 + \frac{1}{n}\right)^{\frac{1}{n-1}} - 1,$$

then the polynomial given by (3.1),

$$f_{h+1}(x) = (q_{h-1} + q_h x)^n f\left(\frac{p_{h-1} + p_h x}{q_{h-1} + q_h x}\right),$$

has at most one variation [18]).

Proof. The first part of the proof partially follows Vincent's original argument. To simplify the notation, we set, as in Section 4, $a = \dfrac{p_{h-1}}{q_{h-1}}$, $b = \dfrac{p_h}{q_h}$, and we make the change of variable $x \leftarrow \dfrac{q_{h-1}}{q_h}x$. We are led to study the number of variations of the polynomial

$$\phi(x) = (1+x)^n f\left(\frac{a+bx}{1+x}\right),$$

the image of f under (4.1).

---

[18]) In Uspensky's original proof [35] one reads $F_{h-1}\Delta > \frac{1}{2}$, probably a misprint that Uspensky had no time to correct, since he died before the publication of the book. The mistake, frequently reproduced, was corrected by Akritas in [3]. But our rereading of Uspensky's proof shows that this hypothesis is unnecessary.

Formulae (3.3) and (3.4) describe the behaviour of the linear and of the quadratic factors of $f(x)$. The hypothesis $F_{h-1}\frac{\Delta}{2} > 1$, which obviously implies the weaker hypothesis $F_h F_{h-1}\Delta > 1$, immediately allows us to prove that no complex root can be transformed into a root having a positive real part, and that at most one real root can be transformed into a positive real root.

Indeed, it follows from $F_h F_{h-1}\Delta > 1$ that

$$|b-a| = \frac{1}{q_h q_{h-1}} < \frac{1}{F_h F_{h-1}} < \Delta,$$

and consequently at most one real root lies in the interval (a,b). A quick look at formula (4.2) allows us to adapt the argument given in Remark 2 to the present situation, in order to exclude that a complex root lies in the circle having the real points a and b as the endpoints of a diameter.

Consider now the roots $x_0, x_1, \dots, x_{n-1}$ of $f(x)$. If no root is in (a,b) then all the factors of the transformed polynomial $\phi(x)$ have positive coefficients, hence $\phi(x)$ has no variations, and the theorem is proved.

Let x_0 be the necessarily unique root of $f(x)$ lying in (a,b), and denote by x_j any other (real or complex) root.

The root x_j is transformed into

$$\xi_j = \frac{x_j-a}{b-x_j} = -1 + \frac{b-a}{b-x_j} = -1+\alpha_j\,.$$

Now $|b-x_j| = |b-x_0+x_0-x_j| \geq |x_0-x_j| - |b-x_0| \geq \Delta - |b-a|$. It follows that

$$|\alpha_j| = \left|\frac{b-a}{b-x_j}\right| \leq \frac{|b-a|}{\Delta-|b-a|}\,.$$

Recalling that $|b-a| = \dfrac{1}{q_h q_{h-1}}$, and that $\Delta F_h F_{h-1} > 1 + \dfrac{1}{\varepsilon_n}$, we conclude that

$$|\alpha_j| < \varepsilon_n\,.$$

The polynomial $\phi(x)$ is of the form

$$(x-\xi_0)(x+1+\alpha_1)(x+1+\alpha_2)\cdot\ldots\cdot(x+1+\alpha_{n-1})\,, \tag{5.4}$$

where $|\alpha_j| < \varepsilon_n$, for $j=1,\dots,n-1$. Let

$$(x+1+\alpha_1)(x+1+\alpha_2)\cdot\ldots\cdot(x+1+\alpha_{n-1}) = x^{n-1}+R_1x^{n-2}+\cdots+R_{n-2}x+R_{n-1}\,.$$

The coefficient R_k is given by the sum of $\binom{n-1}{k}$ products of the form $(1+\alpha_{i_1})(1+\alpha_{i_2})\cdot\ldots\cdot(1+\alpha_{i_k})$, and

$$\begin{aligned}|(1+\alpha_{i_1})(1+\alpha_{i_2})\cdot\ldots\cdot(1+\alpha_{i_k})-1| &\leq (1+|\alpha_{i_1}|)\cdot\ldots\cdot(1+|\alpha_{i_k}|)-1\\ &\leq (1+\varepsilon_n)^k-1\leq(1+\varepsilon_n)^{n-1}-1=\frac{1}{n}.\end{aligned}$$

Hence

$$R_k=\binom{n-1}{k}(1+\delta_k),$$

with

$$|\delta_k|<\frac{1}{n}.$$

Now Lemma 5.1 may be applied to deduce that

$$R_{k+1}^2-R_kR_{k-1}>0,$$

and the argument used to conclude Vincent's proof also ensures that the transformed polynomial has only one variation. □

REMARK 7. In [3], Akritas observes that the last part of this proof is of enough interest to be stated as an independent Lemma:

If a real polynomial of degree $n>1$ *has one positive root, while all the other roots are concentrated in a circular neighbourhood of* -1 *with radius* ε_n, *then the polynomial has exactly one variation.*

In [9], this Lemma is presented as a converse of the rule of signs. Another converse is given by a corollary to Obreschkoff's Lemma presented in Section 8. In any case, the problem is now reduced to that of evaluating an integer h such that the substitution (3.2) sends all the roots but the positive one into a neighbourhood of -1. Uspensky's proof, while ingenious, looks unnecessarily complicated, because the form (5.4) of the transformed polynomial does not reflect the fact that the complex roots of real polynomials appear in conjugate pairs. And instead of looking for a location of the roots ξ_k such that the number of variations does not increase, Uspensky, like Vincent, looks for a polynomial "close" to $(1+x)^{n-1}$. As a consequence he requires that the roots of the transformed polynomial lie in a very small neighbourhood of -1 (of radius ε_n, in fact), which in turn introduces the unnatural condition $F_hF_{h-1}\Delta>1+\dfrac{1}{\varepsilon_n}$. We shall prove that the result holds if $F_hF_{h-1}\Delta>\dfrac{2}{\sqrt{3}}$, and independently of n.

6. VINCENT'S PROOF REVISITED IN MODERN TERMS

The 'qualitative' argument used by Vincent to prove his theorem can easily be recast in modern terms to obtain Uspensky's result, but under the only condition that

$$F_h F_{h-1}\Delta > 1 + \frac{1}{\varepsilon_n}\,.$$

In view of the fact that Viète's formulae relate the coefficients of a polynomial to its roots, it is far from astonishing that Vincent's proof can be improved to provide a quantitative estimate for h. But it is worthwhile to observe that it gives exactly Uspensky's result.

Consider once again the proof of Theorem 4.1 up to (4.3), which describes the polynomial $G(x)$.

Factoring out $g(b)$ we have

$$G(x) = (x+1)^{n-1} g(b)\left[1 + \frac{g'(b)}{1!\cdot g(b)}u + \frac{g''(b)}{2!\cdot g(b)}u^2 + \cdots + \frac{g^{(n-1)}(b)}{(n-1)!\cdot g(b)}u^{n-1}\right].$$

Recalling that $u = \dfrac{a-b}{1+x}$, we have

$$\text{(6.1)}\qquad \frac{G(x)}{g(b)} = (x+1)^{n-1} + \frac{g'(b)}{1!\cdot g(b)}(a-b)(x+1)^{n-2} + \cdots + \frac{g^{(n-1)}(b)}{(n-1)!\cdot g(b)}(a-b)^{n-1}\,.$$

Since the roots of $g(x)$ are $x_1, x_2, \ldots, x_{n-1}$, we have

$$\frac{g'(x)}{1!\cdot g(x)} = \sum_i \frac{1}{1!}\frac{1!}{x-x_i} = \sum_i \frac{1}{x-x_i}\,,$$

$$\frac{g''(x)}{2!\cdot g(x)} = \sum_{i,j} \frac{1}{2!}\frac{2!}{(x-x_i)(x-x_j)} = \sum_{i,j} \frac{1}{(x-x_i)(x-x_j)}\,,$$

$$\frac{g'''(x)}{3!\cdot g(x)} = \sum_{i,j,k} \frac{1}{3!}\frac{3!}{(x-x_i)(x-x_j)(x-x_k)} = \sum_{i,j,k} \frac{1}{(x-x_i)(x-x_j)(x-x_k)}\,,$$

. . . .

The above sums contain respectively $\binom{n-1}{1}$, $\binom{n-1}{2}$, $\binom{n-1}{3}$, ... terms.

Since $F_h F_{h-1}\Delta > 1 + \dfrac{1}{\varepsilon_n} > 1$, we have, in particular, $|b-a| < \Delta$. Hence

$$|b-a| = \theta \cdot \Delta, \quad \text{with} \quad \theta < 1 .$$

Observe that

$$|b-x_i| = |b-x_0+x_0-x_i| > |x_0-x_i| - |b-x_0| > \Delta - \theta\Delta = (1-\theta)\Delta;$$

hence

$$\left|\frac{g^{(k)}(b)}{k! \cdot g(b)}\right| < \binom{n-1}{k} \frac{1}{(1-\theta)^k \Delta^k},$$

and

$$\left|\frac{g^{(k)}(b)}{k! \cdot g(b)}(a-b)^k\right| < \binom{n-1}{k} \frac{1}{\Delta^k} \Delta^k \cdot \frac{\theta^k}{(1-\theta)^k} = \binom{n-1}{k} (\frac{\theta}{1-\theta})^k .$$

Let $\dfrac{\theta}{1-\theta} = \tau$. The absolute value of the coefficient of x^i on the right hand side of (6.1) is smaller than

$$\binom{n-1}{i} + \binom{n-1}{1}\binom{n-2}{i}\tau + \binom{n-1}{2}\binom{n-3}{i}\tau^2 + \binom{n-1}{3}\binom{n-4}{i}\tau^3 + \cdots = \sum_{k=0}^{n-1-i} \binom{n-1}{k}\binom{n-1-k}{i}\tau^k$$
$$= \binom{n-1}{i}(1+\tau)^{n-1-i} = \binom{n-1}{i}(\frac{1}{1-\theta})^{n-1-i} .$$

To apply Lemma 5.1 we need to impose the condition

$$\left|(\frac{1}{1-\theta})^{n-1-i} - 1\right| < \frac{1}{n} \qquad \forall i, \tag{6.2}$$

which is equivalent to

$$\left(\frac{1}{1-\theta}\right)^{n-1} < 1 + \frac{1}{n},$$

that is

$$\theta < 1 - \frac{1}{(1+\frac{1}{n})^{\frac{1}{n-1}}} = \frac{\varepsilon_n}{1+\varepsilon_n} . \tag{6.3}$$

It follows from

$$F_h F_{h-1} \Delta > 1 + \frac{1}{\varepsilon_n}$$

that

$$q_h q_{h-1} \Delta = \frac{\Delta}{|b-a|} = \frac{1}{\theta} > 1 + \frac{1}{\varepsilon_n},$$

hence (6.3) holds and (6.2) is satisfied. Lemma 5.1 may be applied to conclude the proof. □

7. Chen's proof of Vincent's theorem

The proof given by Chen in 1987 [17] has the merit of focusing on the fractional linear transformations of the complex plane into itself as one of the principal tools involved in Vincent's theorem. Keeping the previous notation, we observe that the variable substitution

$$x \leftarrow \frac{a+bx}{1+x}$$

whose effect we have considered in detail, corresponds to the map $\mathcal{F}\colon \mathbf{C} \to \mathbf{C}$ defined by

$$y = \mathcal{F}(x) = \frac{x-a}{b-x}. \tag{7.1}$$

Chen's proof, which also carries over to the case of multiple roots, depends on a careful consideration of the effect of the map (7.1) on the roots of a polynomial.

Another essential tool is given by Obreschkoff's generalization of Descartes' rule of signs which may be stated as follows:

THEOREM 7.1 [30, p. 84]. *The number of roots of a real algebraic equation of degree n with V variations, whose argument φ verifies the inequality*

$$-\frac{\pi}{n-V} < \varphi < \frac{\pi}{n-V},$$

is equal to V or is less than V by an even number.

We list some properties of the map (7.1) we are going to use.

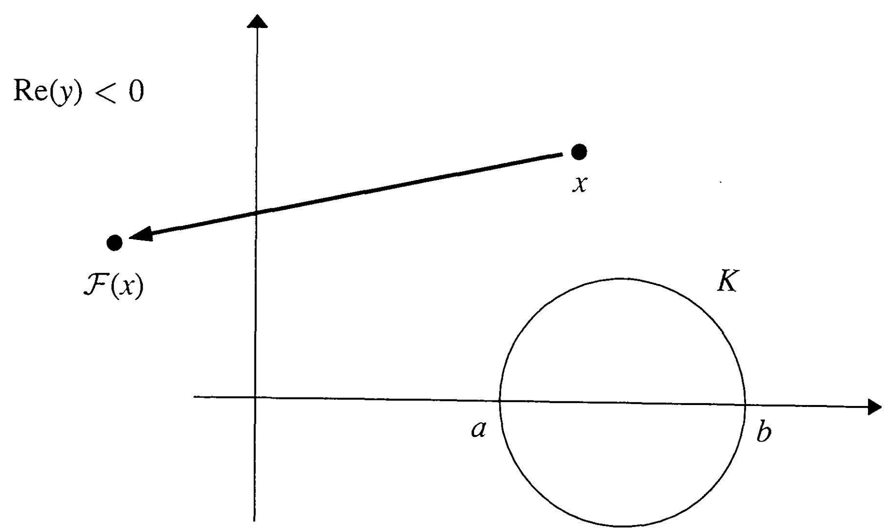

FIGURE 1a

- If $x \in \mathbf{R}$ then $\mathcal{F}(x) \in \mathbf{R}$ and, more precisely, if $x \in (a, b)$ then $\mathcal{F}(x) \in \mathbf{R}^+$.
- The map (7.1) transforms the circle K

$$\left| x - \frac{a+b}{2} \right| = \frac{1}{2} |b-a|$$

into the line $\mathrm{Re}(y) = 0$, and the exterior of this circle into the half-plane $\mathrm{Re}(y) < 0$ (see Fig. 1a).
- The map (7.1) (see Fig. 1b) transforms the left half plane $\mathrm{Re}(x) < 0$ into the interior of the circle H

$$\left| x + \frac{a+b}{2b} \right| = \frac{|a-b|}{2b}$$

whose diameter endpoints are -1 and $-\frac{a}{b}$. The imaginary axis is transformed into H.

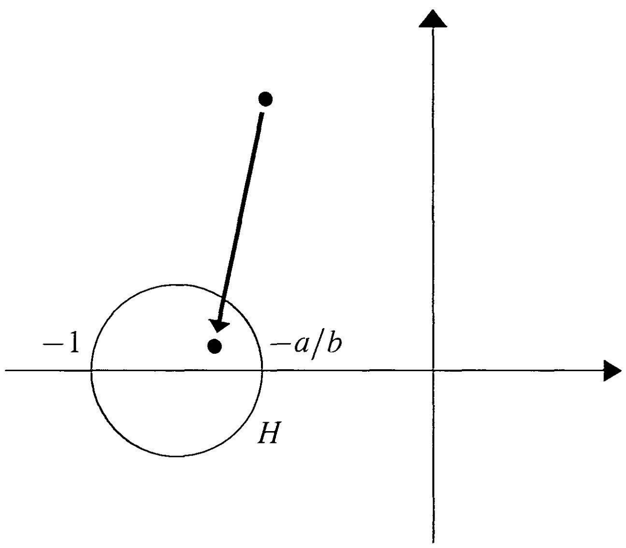

FIGURE 1b

With the help of these observations, we can prove the following

THEOREM 7.2 (Chen's Theorem 1). *Let $f(x)$ be a real polynomial of degree n whose least roots distance is Δ, and let $\gamma = [c_0, c_1, \ldots]$ with c_i non-negative integers, be a continued fraction, whose k-th convergent is denoted by $\frac{p_k}{q_k}$. Suppose that $F_h F_{h-1} \Delta > 1$. Then the polynomial*

$$f_{h+1}(x) = (q_{h-1} + q_h x)^n f\left(\frac{p_{h-1} + p_h x}{q_{h-1} + q_h x}\right)$$

has at most one root in the right half plane: one root if $f_{h+1}(x)$ has a positive number of variations, and no root if it has no variations.

Proof. We set, as before, $a = \frac{p_{h-1}}{q_{h-1}}$, $b = \frac{p_h}{q_h}$ and make the change of variable $x \leftarrow \frac{q_{h-1}}{q_h}x$. Once again we are led to study the number of variations of

$$\phi(x) = (1+x)^n f\left(\frac{a+bx}{1+x}\right).$$

Since

$$|b-a| = \frac{1}{q_h q_{h-1}} \leq \frac{1}{F_h F_{h-1}} < \Delta$$

at most one root of $f(x)$ (which is necessarily real) may be in the interior of K.

If no root is in the interior of K, all the roots are mapped by $\mathcal{F}$ into the left half plane, and $\phi(x)$ has only factors with positive coefficients, and consequently has no variations. If a root x_0 (> 0) is in the interior of K, then $\mathcal{F}(x_0)$ is a positive real number and $\phi(x)$, having a positive root, must have a positive number of variations. $\square$

Chen is now in a position to prove the following theorem.

THEOREM 7.3 (Chen's Theorem 2). *Suppose that the real polynomial $f(x)$ of degree n has only one root x_0 in the right half plane, and consider the continued fraction $\gamma = [c_0, c_1, \ldots]$ with c_i non-negative integers. Suppose that the integer h is sufficiently large to have*

$$\min(p_h q_{h-1}, p_{h-1} q_h) > \frac{n}{6}.$$

If the polynomial $f_{h+1}(x)$ has V variations, then V is exactly the multiplicity of x_0 and $x_0 \in \left(\frac{p_{h-1}}{q_{h-1}}, \frac{p_h}{q_h}\right)$.

Proof. To avoid trivial cases, we suppose that $n \geq 3$ and that $V \geq 3$. We substitute $\phi(x)$ for $f_{h+1}(x)$ as in the previous theorem.

By hypothesis f has only one (real) root x_0 in the right half plane. If $x_0 \notin \left(\frac{p_{h-1}}{q_{h-1}}, \frac{p_h}{q_h}\right)$ then $\phi(x)$ has no variations: hence a contradiction. Therefore, $x_0 \in \left(\frac{p_{h-1}}{q_{h-1}}, \frac{p_h}{q_h}\right)$ and the multiplicity of the root $\mathcal{F}(x_0)$ of $\phi(x)$ is smaller than V. Since $\mathcal{F}$ is one-to-one it follows that the multiplicity of x_0 is also smaller than V.

By the fundamental theorem of algebra the number of roots of $f(x)$ in the left half plane or on the imaginary axis is greater than $n - V$.

$\mathcal{F}$ transforms all the roots different from x_0 into or on the circle H, hence $\phi(x)$ has a positive real root $\mathcal{F}(x_0)$ and all the other roots, whose number is greater than $n-V$, are inside H or on its circumference.

Let $g(x) = \phi(-x)$ and denote by $-H$ the circle symmetric to H with respect to the imaginary axis. The polynomial $g(x)$ has at least $n-V$ roots inside $-H$ or on its circumference. Denoting by V' the number of variations of $g(x)$ we have

$$V' \leq n - V .$$

We prove that the number of the roots of $g(x)$ within $-H$ (or on its boundary) is exactly $n-V$.

From

$$\min(p_{h-1}\, q_h, p_h\, q_{h-1}) > \frac{n}{6} ,$$

we have

$$\frac{1}{2\, p_{h-1}\, q_h} < \frac{3}{n} \quad \text{and} \quad \frac{1}{2\, p_h\, q_{h-1}} < \frac{3}{n} .$$

Hence

$$\frac{3}{n} < \frac{\pi}{n} < \tan\frac{\pi}{n} .$$

It follows that

$$\frac{1}{2\, p_{h-1}\, q_h} < \tan\frac{\pi}{n}, \quad \frac{1}{2\, p_h\, q_{h-1}} < \tan\frac{\pi}{n} .$$

The maximum absolute value of the tangent of the argument of a point inside the circle $-H$ is given by

$$\frac{|a-b|}{2\sqrt{ab}} = \frac{1}{2\sqrt{p_h\, q_{h-1}\, p_{h-1}\, q_h}} ,$$

and

$$\frac{1}{2\sqrt{p_h\, q_{h-1}}}\, \frac{1}{\sqrt{p_{h-1}\, q_h}} \leq \frac{1}{2\min(p_h\, q_{h-1}, p_{h-1}\, q_h)} .$$

It follows that the circle $-H$ is contained in the sector

$$W = \left\{ x \,:\, |\arg(x)| < \frac{\pi}{n} \right\} .$$

The polynomial $g(x)$ has degree n and it has V' variations. Since $V' < n$, we have

$$\frac{\pi}{n-V'} > \frac{\pi}{n} .$$

We may apply Obreschkoff's result to conclude that the number of roots of $g(x)$ within the sector

$$W' = \left\{x : |\arg(x)| < \frac{\pi}{n - V'}\right\}$$

is less than $V' \leq n - V$. But since $g(x)$ has at least $n - V$ roots within or on the boundary of $-H \subseteq W \subseteq W'$, the number of roots of $g(x)$ within or on $-H$ is exactly $n - V$. Then $\phi(x)$ and therefore $f_{h+1}(x)$ has exactly $n - V$ roots in the left half plane. Hence V is the multiplicity of the only positive root of $f_{h+1}(x)$ and therefore of $f(x)$. □

THEOREM 7.4. *Let $f(x)$ be an integral polynomial of degree $n \geq 3$, with only one root x_0 in the right half plane, and suppose it has at least 3 variations. Let m be the smallest integer such that*

$$m > \tfrac{1}{2} \log_{\phi} n,$$

where $\phi = \frac{1+\sqrt{5}}{2}$. Let $\gamma = [c_0, c_1, \ldots]$ with c_i positive integers. If V is the number of variations of f_{m+1}, then the root x_0 has multiplicity V and $x_0 \in \left(\frac{p_{m-1}}{q_{m-1}}, \frac{p_m}{q_m}\right)$.

Proof. Since

$$m > \tfrac{1}{2} \log_{\phi} n,$$

we have

$$\phi^{2m} > n.$$

Let $\psi = \frac{1-\sqrt{5}}{2}$. Writing the n-th Fibonacci number F_n in our notation (see note 4) as

$$F_n = \frac{1}{\sqrt{5}}(\phi^{n+1} - \psi^{n+1}),$$

we easily deduce

$$F_{m-1}^2 \geq \frac{n}{6}.$$

The hypothesis $c_i \geq 1$ implies that $p_k > q_k$ for every k, hence

$$\min(p_m q_{m-1}, p_{m-1} q_m) > F_{m-1}^2 \geq \frac{n}{6}$$

and we may apply the previous theorem [19]). □

[19]) The reason why Chen does not explicitly require $F_{m-1}^2 > n/6$ is not clear, but we have followed his approach.

THEOREM 7.5 (Chen's Main Theorem). *Let $f(x)$ be an integral polynomial of degree $n \geq 3$ with at least 3 variations. Let h be the smallest positive integer for which*

$$F_{h-1}^2 \Delta > 1 ,$$

and let m be the smallest positive integer such that

$$m > \tfrac{1}{2} \log_\phi n .$$

Let $k = h + m$. For an arbitrary continued fraction $\gamma = [c_0, c_1, \ldots] > 0$, consider the polynomial f_{k+1} constructed by F_k. If V is the number of variations of f_{k+1} then the polynomial f has a unique positive root in $\left(\frac{p_{k-1}}{q_{k-1}}, \frac{p_k}{q_k}\right)$ and V is its multiplicity.

Proof. After h steps, the polynomial f_{h+1} might have no variations, and then f_{k+1} will have no variations. If f_{h+1} has V variations, by Chen's Theorem 1 it has a positive root in the right half plane. The partial quotients c_i are ≥ 1 for $i > h$, and so we may apply Chen's Theorem 2. □

8. A NEW PROOF OF VINCENT'S THEOREM

In this section we give a new and simpler proof of Vincent's theorem, which in turn improves on Chen's result. For the sake of clarity, we prefer to deal separately with the two cases of simple and multiple roots.

8.1 THE CASE OF SIMPLE ROOTS

In the case of simple roots, we show that Vincent's theorem holds under the only assumption

$$\Delta F_h F_{h-1} > \frac{2}{\sqrt{3}}$$

independently of the polynomial degree n.

Our proof depends on the following result by Obreschkoff [30, p. 81].

LEMMA 8.1. *Let $f(x)$ be a real polynomial with V variations in the sequence of its coefficients; let V_1 be the number of variations of the polynomial $f_1(x) = (x^2 + 2\rho x \cos\varphi + \rho^2) f(x)$ (where $\rho > 0$ and $|\varphi| < \frac{\pi}{V+2}$). Then $V \geq V_1$, and the difference $V - V_1$ is an even number.*

The analogous result, for the polynomial $(x+p)f(x)$, with $p \in \mathbf{R}^+$, can also be proved by a slight modification of Obreschkoff's proof. In the sequel, this extended version comprising both cases, will be referred to as Lemma 8.1.

For $V = 1$ we get the following

COROLLARY 8.2. *If a real polynomial has one positive simple root x_0, and all the other (possibly multiple) roots lie in the sector:*

$$S_{\sqrt{3}} = \{x = -\alpha + i\beta \mid \alpha > 0 \text{ and } \beta^2 \leq 3\alpha^2\}$$

then the sequence of its coefficients has exactly one sign variation.

Obreschkoff argued by contradiction. We give here a simple constructive proof of the corollary.

Proof. It suffices to prove that if

$$f(x) = \sum_{k=0}^{n} a_k x^{n-k}$$

is a real polynomial whose sequence of coefficients has one variation, then both

$$f_\alpha(x) = (x+\alpha)f(x),$$
$$f_{\alpha,\beta}(x) = \left[(x+\alpha)^2 + \beta^2\right] f(x)$$

have exactly one variation. Starting with $f(x) = (x - x_0)$ and iterating the above argument for any root in $S_{\sqrt{3}}$ one gets the claim.

Indeed,

$$f_\alpha(x) = \sum_{k=0}^{n+1} b_k x^{n+1-k}$$

where (setting $a_{-1} = a_{n+1} = 0$)

$$b_k = a_k + \alpha\, a_{k-1} \quad \text{for } k = 0, 1, \ldots, n+1\,.$$

If f has one variation, then there exist indices j and i, with $0 \leq j < i \leq n$, such that

$$\begin{aligned} &a_{-1}, a_0, \ldots, a_{j-1} \geq 0 \quad \text{and} \quad a_j > 0\,, \\ (8.1) \qquad &a_{j+1} = \cdots = a_{i-1} = 0 \quad (\text{if } i > j+1)\,, \\ &a_i < 0 \quad \text{and} \quad a_{i+1}, \ldots, a_{n+1} \leq 0\,, \end{aligned}$$

whence

$$b_0, \dots, b_{j-1} \geq 0 \quad \text{and} \quad b_j > 0\,,$$
$$b_{i+1} < 0 \quad \text{and} \quad b_{i+2}, \dots, b_{n+1} \leq 0\,.$$

If $i = j+1$ only the sign of b_i is unpredictable; if $i \geq j+2$, then $b_{j+1} > 0$ and $b_i < 0$ (and $b_{j+2} = \cdots = b_{i-1} = 0$ if $i > j+2$). In any case, the polynomial f_α has just one variation.

Now consider

$$f_{\alpha,\beta}(x) = \sum_{k=0}^{n+2} d_k\, x^{n+2-k} \qquad \text{where} \quad d_k = a_k + 2\,\alpha\, a_{k-1} + (\alpha^2 + \beta^2)\, a_{k-2}$$

with

$$a_{-2} = a_{-1} = 0 = a_{n+1} = a_{n+2}\,.$$

If (8.1) still holds (including $a_{-2} = a_{n+2} = 0$), then

$$d_0, \dots, d_{j-1} \geq 0 \text{ and } d_j > 0 \ \dots \text{ and } d_{i+2} < 0 \text{ and } d_{i+3}, \dots, d_{n+2} \leq 0\,.$$

If $i \geq j+3$ or $i = j+2$, then the sequence d_k has one variation.

If $i = j+1$ we show that

$$d_{j+1} < 0 \quad \text{implies} \quad d_{j+2} \leq 0$$

and this suffices to prove that the sequence d_k has only one variation.

The inequality

$$d_{j+1} = a_{j+1} + 2\,\alpha\, a_j + (\alpha^2 + \beta^2) a_{j-1} < 0$$

with $a_{j-1} \geq 0$, implies $a_{j+1} + 2\alpha a_j < 0$. Therefore

$$\begin{aligned} d_{j+2} &= a_{j+2} + 2\,\alpha\, a_{j+1} + (\alpha^2 + \beta^2)\, a_j \\ &\leq 2\,\alpha\, a_{j+1} + (\alpha^2 + 3\alpha^2)\, a_j = 2\,\alpha(a_{j+1} + 2\,\alpha\, a_j) \leq 0 \end{aligned}$$

(since $a_{j+2} \leq 0$) and this completes the proof. □

Now we prove the theorem in our stronger form.

Proof. We use the same notation as before. Suppose h is such that

$$\Delta F_h F_{h-1} > \frac{2}{\sqrt{3}}\,.$$

Since $\dfrac{2}{\sqrt{3}} > 1$, we know by the previous argument that all the quadratic irreducible factors of the transformed polynomial $\phi(x)$ have nonnegative coefficients and that at most one linear factor of $\phi(x)$ has coefficients with opposite signs. This happens if and only if there is a positive root x_0 of $f(x)$, which belongs to the interval (a,b).

We only need to show that all the roots of $f(x)$ different from x_0 are mapped into the sector $S_{\sqrt{3}}$. In this case the transformed polynomial $\phi(x)$ has exactly one variation by Corollary 8.2.

Once again we consider the map $\mathcal{F}\colon \mathbf{C} \to \mathbf{C}$ defined by

$$y = \mathcal{F}(x) = \frac{x-a}{b-x}\,. \tag{8.2}$$

In the previous section we observed that the map (8.2) transforms the circle

$$\left|x - \frac{a+b}{2}\right| = \frac{1}{2}|b-a| \tag{8.3}$$

into the line $\mathrm{Re}(y) = 0$, and the exterior of this circle into the half-plane $\mathrm{Re}(y) < 0$ (see Fig. 1a).

But another property of (8.2) is relevant: the circles passing through the points a, b are sent into lines through the origin of the complex plane. More precisely, the lines

$$\mathrm{Im}(y) = \pm s\,\mathrm{Re}(y) \quad (s \in \mathbf{R}^+)$$

are the images of the circles centered at

$$c^{\pm} = \frac{a+b}{2} \pm i\frac{|b-a|}{2s}\,,$$

with radius

$$r = \frac{|b-a|}{2}\sqrt{1+\frac{1}{s^2}}\,.$$

It easily follows (see Fig. 2) that the sector S of the complex plane defined by

$$\mathrm{Re}(y) < 0 \quad \text{and} \quad |\mathrm{Im}(y)| \leq |s| \cdot |\mathrm{Re}(y)| \tag{8.4}$$

is the image under $\mathcal{F}$ of the exterior of the eight-shaped figure R given by the union of the two disks

$$\left|x - c^{\pm}\right| \leq r\,.$$

A point x at distance greater than $2r$ from a point of the segment with endpoints a, b cannot be in the interior of R, and hence $\mathcal{F}(x) \in S$.

To ensure the existence of at most one variation in the transformed polynomial, we must require that $s = \sqrt{3}$, and hence consider the particular sector $S_{\sqrt{3}}$.

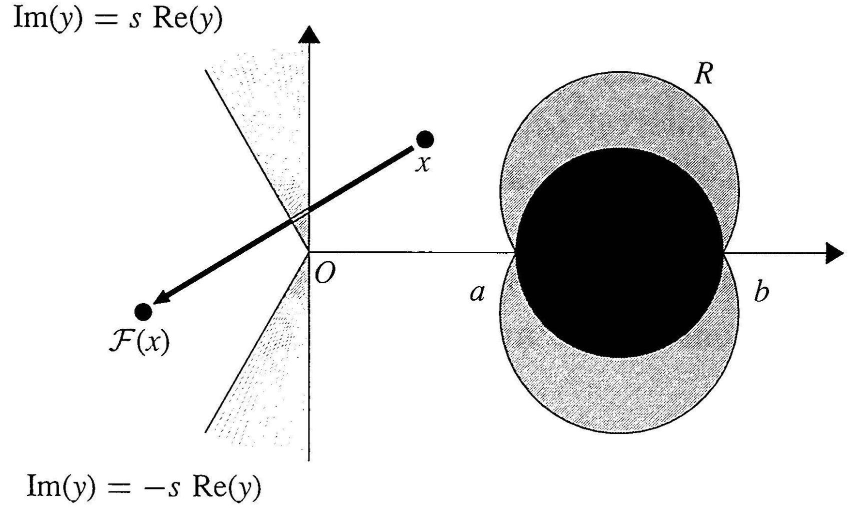

FIGURE 2

Now $r = \dfrac{|b-a|}{\sqrt{3}}$ and $\Delta F_h F_{h-1} > \dfrac{2}{\sqrt{3}}$ imply

$$\Delta q_h q_{h-1} > \frac{2}{\sqrt{3}},$$

that is

$$\frac{\Delta}{|b-a|} > \frac{2}{\sqrt{3}} \quad \text{or} \quad \Delta > 2r.$$

Only the root x_0 is in the interior of R, and so any other root is mapped by (8.2) into S. □

REMARK 8. To compute the continued fraction expansions of algebraic numbers which occur as zeros of integer polynomials, an interesting class of polynomials is given by the *reduced polynomials* (see [12] and [14]). A polynomial $f(x)$ is *reduced* if it has a unique root $x_0 > 1$ and all its other roots x_j satisfy $|x_j| < 1$ and $\mathrm{Re}(x_j) < 0$. A reduced polynomial does not necessarily have a unique variation, nor is a polynomial with a unique variation necessarily reduced. But it is interesting to observe that the machinery (3.1) of Vincent's theorem establishes a deep connection between the two classes of polynomials. In [12], the authors give a brilliant proof that, for sufficiently large h, the polynomial f_h is reduced. A remarkable difference between reduced polynomials and polynomials with a single variation is that we can immediately check that a polynomial has a single variation, while it is not so immediate to verify that a polynomial is reduced. A possible test is given by the combined use of Theorems 40.2 and 42.1 of [29]. Since we have replaced the transformation

$$x \leftarrow \frac{p_{h-1} + p_h x}{q_{h-1} + q_h x}$$

by

$$x \leftarrow \frac{a + bx}{1 + x},$$

we have to replace the unitary circle by the circle of radius $\frac{q_h}{q_{h-1}}$. We obtain a reduced polynomial if we require that under the map (8.2) the image $\mathcal{F}(x_j) = \frac{x_j - a}{b - x_j}$ of a root x_j of $f(x)$, different from x_0, is such that

(8.5) $$|\mathcal{F}(x_j)| < \frac{q_h}{q_{h-1}}, \qquad \text{and} \qquad \operatorname{Re} \mathcal{F}(x_j) < 0\,.$$

Let $t = \frac{q_h}{q_{h-1}}$, and consider the Apollonius circle

(8.6) $$\left| \frac{x - a}{x - b} \right| = t\,.$$

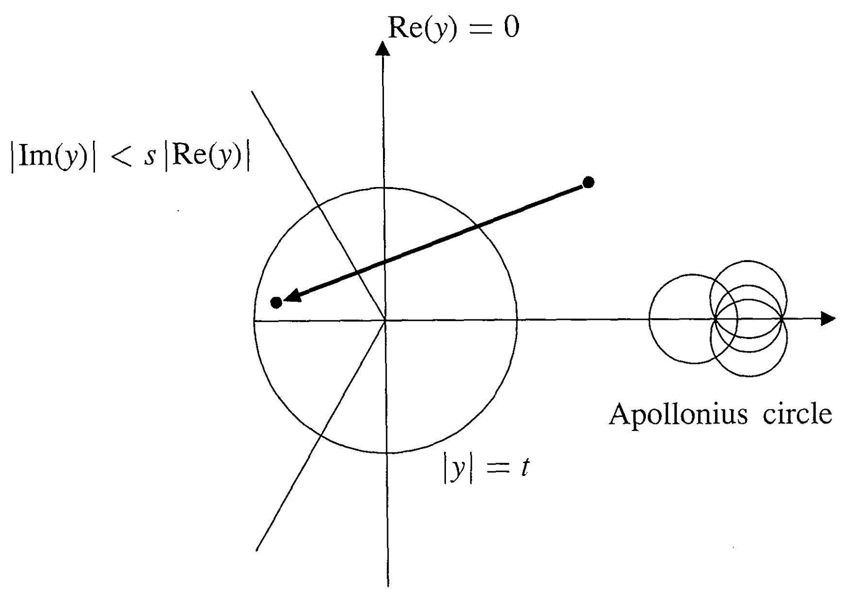

FIGURE 3

$\mathcal{F}$ maps the exterior of the circle (8.6) into the interior of the circle $|y| = t$. Hence the first condition in (8.5) means that x_j must be outside the circle (8.6). The diameter of the circle (8.6) lies on the real axis, and its endpoints are

$$u = \frac{a - tb}{1 - t}, \quad v = \frac{a + tb}{1 + t}\,.$$

Clearly $v > 0$; moreover

$$u = \frac{\dfrac{p_{h-1}}{q_{h-1}} - \dfrac{q_h}{q_{h-1}}\dfrac{p_h}{q_h}}{1 - \dfrac{q_h}{q_{h-1}}} = \frac{p_{h-1} - p_h}{q_{h-1} - q_h} > 0 .$$

It follows that this circle is entirely contained in the right half plane. A root x_j different from x_0 (see Fig. 3) lies outside the circle (8.3) if h is large enough to have $\Delta > |b-a|$. It follows that $\operatorname{Re}\mathcal{F}(x_j) < 0$, and hence it is external to the circle (8.6) corresponding to the value $h+1$. Hence the condition

$$F_{h-1}F_{h-2}\,\Delta > 1$$

ensures that the polynomial f_{h+1} is reduced.

8.2 The case of multiple roots

Obreschkoff's Lemma 8.1 yields the following

COROLLARY 8.3. *Let $f(x) = (x-x_1)(x-x_2)\cdot\ldots\cdot(x-x_r)$, where $x_i \in \mathbf{R}^+$. Then*

$$f_1(x) = (x^2 + 2\,\rho\, x\cos\varphi + \rho^2)f(x)\,, \quad \rho > 0\,, \quad |\varphi| < \frac{\pi}{r+2}$$

has exactly r variations. More generally, a polynomial having r positive real roots and all its other roots in the sector

$$S = \left\{x \;\middle|\; x = -\rho(\cos\varphi + i\sin\varphi), \quad \rho > 0, \quad |\varphi| < \frac{\pi}{r+2}\right\}$$

has exactly r variations.

This allows us to extend Vincent's theorem to the case of multiple roots. Suppose the polynomial $f(x)$ has multiple roots, and let Δ be their least distance. If h is sufficiently large to verify

$$F_hF_{h-1}\,\Delta > 1\,,$$

at most one root x_0 lies in (a,b), but since this root may have multiplicity r, f_h has 0 or at least r variations. It will have exactly r variations if we can ensure that $x_0 \in (a,b)$ and that the other transformed roots lie in the sector

$$S = \{y \mid \operatorname{Re} y < 0, \quad |\operatorname{Im} y| < |\tan\varphi|\cdot|\operatorname{Re} x|\}\ ,$$

where $\varphi = \dfrac{\pi}{r+2}$. Let $s = \tan\dfrac{\pi}{r+2}$ and let us make the appropriate substitutions into (8.4). We have proved

THEOREM 8.4. *Let $f(x)$ be a real polynomial of degree n whose roots are of multiplicity smaller than r. Let $\gamma = [c_0, c_1, c_2, \dots]$ and, maintaining the previous notation, consider the polynomials*

$$f_{h+1} = (q_{h-1} + q_h x) f\left(\frac{p_{h-1} + p_h x}{q_{h-1} + q_h x}\right).$$

Let $s = \tan \dfrac{\pi}{r+2}$. *If h satisfies*

$$F_h F_{h-1} \Delta > \sqrt{1 + \frac{1}{s^2}} = \frac{1}{\sin \frac{\pi}{r+2}},$$

then the number of variations of f_{h+1} equals the multiplicity of the root in $\left(\dfrac{p_{h-1}}{q_{h-1}}, \dfrac{p_h}{q_h}\right)$.

REMARK 9. Obviously, letting $s = \tan \frac{\pi}{n+2}$, we can implement an algorithm to isolate the roots, without being forced to reduce the polynomial $f(x)$ to one with simple roots.

REMARK 10. We conclude our paper by showing that our estimate of the size of h is asymptotically better than Chen's. Suppose we consider a polynomial whose roots are of multiplicity $\leq r$ (which necessarily has degree $n \geq r$). We have proved that the isolation of a root can be carried out in p steps, where p verifies

$$F_p F_{p-1} \Delta > \sqrt{1 + \frac{1}{\tan^2 \frac{\pi}{r+2}}} = \frac{1}{\sin \frac{\pi}{r+2}}. \tag{8.7}$$

We want to compare this integer with that needed by Chen's theorem, that is the smallest integer $m = h + k$, where h and k satisfy

$$F_h F_{h-1} \Delta > 1 \quad \text{and} \quad k > \tfrac{1}{2} \log_\phi r. \tag{8.8}$$

We know that

$$\sqrt{5} F_k \approx \phi^{k+1},$$

hence (8.7) becomes

$$\frac{\Delta}{5} \phi^{2p+1} > \frac{1}{\sin \frac{\pi}{r+2}}.$$

On the other hand, by (8.8) we have

$$\frac{\Delta}{5} \phi^{2h+1} > 1 \quad \text{and} \quad k > \tfrac{1}{2} \log_\phi r. \tag{8.9}$$

The second equality may be rewritten as

$$\phi^{2k} > r\,. \tag{8.10}$$

From the first inequality of (8.9) and (8.10) it follows that

$$\frac{\Delta}{5}\,\phi^{2h+1}\phi^{2k} = \frac{\Delta}{5}\,\phi^{2(h+k)+1} > r\,.$$

Hence

$$\frac{\Delta}{5}\,\phi^{2m+1} > r\,.$$

Since

$$r > \frac{1}{\sin\frac{\pi}{r+2}} \quad \text{for } r \geq 2\,,$$

$m \geq p$ for r sufficiently large and the proof is concluded. □

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REFERENCES

[1] AKRITAS, A. G. Vincent's theorem in algebraic manipulation. Ph. D. Thesis, Operation Research Program, North Carolina State University, Raleigh, N.C., 1978.

[2] —— A new method for polynomial real root isolation[20]. *Proceedings of the 16th annual southeast regional ACM conference*, Atlanta, Georgia, April 1978, 39–43.

[3] —— A correction on a theorem by Uspensky. *Bull. Soc. Math. Grèce (N.S.) 19* (1978), 278–285.

[4] —— Reflections on a pair of theorems by Budan and Fourier. *Math. Mag. 55* (1982), 292–298.

[5] —— There is no "Uspensky's method". Extended abstract. *Proceedings of the 1986 Symposium on Symbolic and Algebraic Computation,* Waterloo, Ontario, Canada, 1986, 88–90.

[6] —— The role of the Fibonacci sequence in the isolation of the real roots of polynomial equations. *Applications of Fibonacci Numbers,* vol. 3, 1988.

[7] —— *Elements of Computer Algebra with Applications.* John Wiley & Sons, New York, 1989.

[8] AKRITAS, A. G. and S. D. DANIELOPOULOS. On the forgotten theorem of Mr. Vincent. *Historia Math. 5* (1978), 427–435.

[9] AKRITAS, A. G. and S. D. DANIELOPOULOS. A converse rule of signs for polynomials. *Computing 34* (1985), 283–286.

[20]) This paper won the first prize in the student paper competition.

[10] AHLFORS, L. V. *Complex Analysis* (2nd ed.). McGraw-Hill, New York, 1966.

[11] BARTOLOZZI, M. and R. FRANCI. La regola dei segni dall'enuciato di R. Descartes (1637) alla dimostrazione di Gauss (1828). *Arch. Hist. Exact Sci. 45* (1993), 335–374.

[12] BOMBIERI, E. and A. J. VAN DER POORTEN. Continued fractions of algebraic numbers. *Computational algebra and number theory,* Sydney, 1992, 137–152. *Math. Appl. 325*, Kluwer Acad. Publ., Dordrecht, 1995.

[13] BOURDON, L. P. M. *Élémens d'Algèbre.* Bachelier père et fils, Paris, 1831, sixième édition.

[14] BRENT, R. P., A. J. VAN DER PORTEN and H. J. J. TE RIELE. A comparative study of algorithms for computing continued fractions of algebraic numbers. In: *Algorithmic Number Theory* (H. Cohen, ed.), Second International Symposium ANTS–II, Talence, France, May 18–23, 1996, 35–47.

[15] BUDAN, F. D. *Nouvelle méthode pour la résolution des équations numériques d'un degré quelconque.* Chez Courcier, Paris, 1807.

[16] CANTOR, D. G., P. H. GALYEAN and H. G. ZIMMER. A continued fraction algorithm for real algebraic numbers. *Math. Comp. 26* (1972), 785–791.

[17] CHEN, J. A new algorithm for the isolation of real roots of polynomial equations. *Second International Conference on Computers and Applications*, Beijing, People's Republic of China, 1987, 714–719, IEEE Computer Soc. Press.

[18] COLLINS, G. E. and A. G. AKRITAS. Polynomial real root isolation using Descartes' rule of signs. *Proceedings of the 1976 ACM Symposium on Symbolic and Algebraic Computation*, 272–275. Yorktown Heights, NY, 1976.

[19] DESCARTES, R. *Œuvres.* Originally edited by C. Adam and P. Tannery, 12 vols. New presentation, Vrin, Paris, 1974–1986.

[20] FOURIER, J. *Analyse des équations déterminées.* Firmin Didot frères libraires, Paris, 1831.

[21] LLOYD, E. KEITH On the forgotten Mr. Vincent. *Historia Math. 6* (1979), 448–450.

[22] LAGRANGE, J. L. *Œuvres de Lagrange.* 14 vols., Gauthier-Villars, Paris, 1867–1892.

[23] —— Sur la résolution des équations numériques. *Histoire de l'Académie royale des sciences et belles-lettres* (Berlin), *23* (1767), 1769, 311–352; *Œuvres 2*, 539–578.

[24] —— Additions au mémoire sur la résolution des équations numériques. *Histoire de l'Académie royale des sciences et belles-lettres* (Berlin) *24* (1768), 1770, 111–180. *Œuvres 2*, 581–652.

[25] —— *De la résolution des équations numériques de tous les degrés.* Duprat, Paris, 1798.

[26] —— *Traité de la résolution des équations numériques de tous les degrés, avec des notes sur plusieurs points de la théorie des équations algébriques.* Enlarged edition of [25], chez Courcier, Paris, 1808. *Œuvres 8.*

[27] LANG, S. and H. TROTTER. Continued fractions for some algebraic numbers. *J. reine angew. Math. 255* (1972), 112–134. Addendum, *ibid.*, 219–220.

[28] LÜTZEN, J. *Joseph Liouville.* Springer, New York, 1990.

[29] MARDEN, M. *The Geometry of the Zeros of a Polynomial in a Complex Variable.* American Mathematical Society, New York, 1949.

[30] OBRESCHKOFF, N. *Verteilung und Berechnung der Nullstellen reeller Polynome.* VEB Deutscher Verlag der Wissenschaften, Berlin, 1963.

[31] POGGENDORFF, J. C. *Biographisch-literarisches Handwörterbuch zur Geschichte der exacten Wissenschaften.* J. A. Barth, Leipzig, 1863.

[32] ROSEN, D. and J. SHALLIT. A continued fraction algorithm for approximating all real polynomial roots[21]). *Math. Mag. 51* (1978), 112–116.

[33] SINACEUR, H. *Corps et modèles.* Vrin, Paris, 1991.

[34] SMITH, D. E. and M. L. LATHAM (eds.). *The Geometry of René Descartes.* Dover publications, New York, 1954.

[35] USPENSKY, J. V. *Theory of Equations.* McGraw-Hill, New York, 1948.

[36] VINCENT, A. J. H. Sur la résolution des équations numériques. *Mémoires de la Société royale de Lille* (1834), 1–34. Also in *Journal de mathématiques pures et appliquées 1* (1836), 341–372.

[37] —— Addition à une précédente Note relative à la résolution des équations numériques. *Mémoires de la Société royale de Lille* (1838), 5–24. Also in *Journal de mathématiques pures et appliquées 3* (1838), 235–243.

[38] WANG, Xianghao. A method for isolating roots of algebraic equations. Jilin Univ. Academic Press, N. 1, 1960.

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[21]) Presented at the *Conference on Computation in Algebra and Number Theory*, Univ. of New Brunswick, Canada, August 1975.