1. Introduction

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **44 (1998)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

L'Enseignement Mathématique, t. 44 (1998), p. 219–256

A NEW PROOF OF VINCENT'S THEOREM

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ABSTRACT. Vincent's theorem (1836) asserts that, given a real polynomial $f(x)$ without multiple roots, the substitution

$$x \leftarrow c_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \ddots + \cfrac{1}{c_h + \cfrac{1}{x}}}}$$

where the c_i are arbitrary positive integers and h is sufficiently large, transforms $f(x)$ into a polynomial $f_{h+1}(x)$ which has at most one sign variation in the sequence of its coefficients.

This theorem is basic for highly efficient methods (implemented in modern computer algebra systems) to separate the roots of a real polynomial.

In this paper we provide a new simple proof of the theorem, which improves the known estimates of the size of h and can be extended to the case of multiple roots. We also give an historical survey of the subject.

1. INTRODUCTION

The aim of this paper is to give a new and simple proof of Vincent's theorem. The theorem has an interesting history.

It originally appeared as a note, *Sur la résolution des équations numériques*, appended at the end of the sixth edition of Bourdon's *Élémens d'algèbre* [13], without explicit mention of Vincent's authorship. Bourdon, who was Vincent's father-in-law [1]), merely acknowledges his debt to his son-in-law for "plusieurs améliorations de détail et quelques additions" in the *Avertissement* at the beginning of his book.

[1]) Information about Vincent, who was an influential personality in his time, can be found in [21] and [31].

The debt must have been important, because Vincent later published the result under his name alone: first in the *Mémoires de la Société royale de Lille* (1834), and afterwards, with some improvements, in the *Journal de mathématiques pures et appliquées* (1836) (see [36]).

Unfortunately (for Vincent), Sturm's theorem concerning the number of real roots of an algebraic equation in a given interval, which originally appeared without proof in 1829 and was then published in complete form in 1835, was growing in popularity and ended by superseding Vincent's result. And times were not ripe to understand the remarkable algorithmic potentialities of Vincent's theorem in comparison with Sturm's (see [7]).

Liouville introduces the publication of Vincent's note in his *Journal* with the unflattering remark that the note was being published again, with some additions to the version which had previously appeared in the *Mémoires de Lille*, "dans l'intérêt des professeurs" [36, p. 341, note]. After a subsequent careful reading of Vincent's paper, Liouville commented[2]: "We do not see that these results, curious as they may be, can be of use in our current research."

The theorem was forgotten until 1948, when it was published in Uspensky's book [35]. Uspensky was the first to describe an algorithm based on Vincent's theorem to separate the roots of a polynomial. But to avoid useless calculations, he didn't follow Vincent's original approach (through Budan's theorem), as was pointed out by Akritas ([3], [5]), who also corrected an error in Uspensky's theorem.

Uspensky, who probably doubted that Vincent's original argument could be turned into a proof satisfying modern standards, elaborated another ingenious, but unnecessarily complicated, proof. In Section 6 we show that the essence of Uspensky's result can be obtained through a careful consideration of Vincent's proof.

After Uspensky's book, the theorem appeared in Obreschkoff's book [30], but without any particular application.

The first implementation of an algorithm based on Vincent's theorem in terms suitable for modern computer algebra was made by Akritas (see [1]) and by Rosen and Shallit ([32], see also [18]). Since then, the considerable attention devoted to the subject by Akritas ([3], [5], [6], [7], [8], [9]) has given this algorithm its present status of a powerful tool of computer algebra systems.

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[2]) Quoted in [28, p. 521]. Liouville's text is in a notebook (Ms 3617 (7)) at the Institut de France (Bibliothèque) in Paris. Quite obviously Liouville does not refer only to the content of Vincent's theorem, but to the possibility of using Vincent's result for the studies about transcendental numbers he was conducting at that time.

Curiously, all the proofs before that of Chen-Wang [17], in 1987, have not really used the fact that the complex roots of a real polynomial appear in conjugate pairs. Nor have they considered the effect of the maps of the complex plane into itself, which are naturally related to Vincent's theorem. Chen's proof, which also depends on Obreschkoff's generalization of Descartes' rule of signs, only partially exploits the consideration of the fractional linear transformations connected to Vincent's Theorem, and is rather complicated [3]).

Only Bombieri and van der Poorten consider in full clarity [12] the behaviour of the roots of a polynomial under the action of the fractional linear transformations related to the problem. Proposition 3.1 of [12] gives a result strictly related to Vincent's theorem, regarding the possibility of obtaining *reduced polynomials* (see Remark 8) instead of polynomials having a single sign variation, but the proof can easily be adapted to the situation of Vincent's theorem.

Our proof of the theorem was inspired by the geometric treatment in [12], and combines the use of geometrical transformations with another result of Obreschkoff [30, III, § 17] for which, in a particular but relevant case, we provide a new direct proof.

The resulting proof of Vincent's theorem is simple and short (to us), and can easily be extended to the case of multiple roots [4]).

2. PRELIMINARY FACTS

As we shall deal extensively with sign variations, we begin with

DEFINITION 2.1. Given a sequence (finite or infinite) of real numbers $\alpha_0, \alpha_1, \alpha_2, \ldots$, we say that there is a *sign variation* between two terms α_p and α_q if one of the following conditions holds:

1) $q = p+1$ and α_p and α_q have opposite signs;

2) $q > p+1$ and the terms $\alpha_{p+1}, \alpha_{p+2}, \ldots, \alpha_{q-1}$ are all zero while α_p and α_q have opposite signs [5]).

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[3]) Unfortunately, we haven't yet been able to get Wang's paper [38], and all our information depends on Chen's paper [17]. Hence we refer to Chen-Wang's theorem.

[4]) For the convenience of the reader, we have decided to unify the notation and the symbolism of a subject which, in more than a century and a half, has been considered in very different forms. Throughout the paper the sequence of Fibonacci numbers $F_0, F_1, \ldots$ begins with 1 instead of 0. Some minor changes have been introduced in the statement as well as in the proof of many theorems to conform to this convention.

[5]) Cf. [7, p. 338]