2. Preliminary facts

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Curiously, all the proofs before that of Chen-Wang [17], in 1987, have not really used the fact that the complex roots of a real polynomial appear in conjugate pairs. Nor have they considered the effect of the maps of the complex plane into itself, which are naturally related to Vincent's theorem. Chen's proof, which also depends on Obreschkoff's generalization of Descartes' rule of signs, only partially exploits the consideration of the fractional linear transformations connected to Vincent's Theorem, and is rather complicated[3]).

Only Bombieri and van der Poorten consider in full clarity [12] the behaviour of the roots of a polynomial under the action of the fractional linear transformations related to the problem. Proposition 3.1 of [12] gives a result strictly related to Vincent's theorem, regarding the possibility of obtaining *reduced polynomials* (see Remark 8) instead of polynomials having a single sign variation, but the proof can easily be adapted to the situation of Vincent's theorem.

Our proof of the theorem was inspired by the geometric treatment in [12], and combines the use of geometrical transformations with another result of Obreschkoff [30, III, § 17] for which, in a particular but relevant case, we provide a new direct proof.

The resulting proof of Vincent's theorem is simple and short (to us), and can easily be extended to the case of multiple roots[4]).

2. Preliminary facts

As we shall deal extensively with sign variations, we begin with

DEFINITION 2.1. Given a sequence (finite or infinite) of real numbers $\alpha_0, \alpha_1, \alpha_2, \ldots$, we say that there is a *sign variation* between two terms α_p and α_q if one of the following conditions holds:

1) $q = p + 1$ and α_p and α_q have opposite signs;

2) $q > p + 1$ and the terms $\alpha_{p+1}, \alpha_{p+2}, \ldots, \alpha_{q-1}$ are all zero while α_p and α_q have opposite signs[5]).

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[3]) Unfortunately, we haven't yet been able to get Wang's paper [38], and all our information depends on Chen's paper [17]. Hence we refer to Chen-Wang's theorem.

[4]) For the convenience of the reader, we have decided to unify the notation and the symbolism of a subject which, in more than a century and a half, has been considered in very different forms. Throughout the paper the sequence of Fibonacci numbers $F_0, F_1, \ldots$ begins with 1 instead of 0. Some minor changes have been introduced in the statement as well as in the proof of many theorems to conform to this convention.

[5]) Cf. [7, p. 338]

Given an arbitrary real polynomial

$$\alpha_0 x^n + \alpha_1 x^{n-1} + \alpha_2 x^{n-2} + \cdots + \alpha_n ,$$

the expression *(sign) variation of the polynomial* will be used as an abbreviation to mean a sign variation in the sequence of its coefficients.

EXAMPLE 2.2. The polynomial $x^7 - 7x^3 + 3x^2 + 5$, whose sequence of coefficients is $\{1,0,0,0,-7,3,0,5\}$, has two variations, while the polynomial $x^5 - 1$ has one variation.

The idea of relating the number of sign variations of a real polynomial to the number of its positive real roots goes back to the beginning of modern algebra. In his *Géométrie* (1637) Descartes boldly writes[6]) (without any trace of a proof): "An equation can have as many true [positive] roots as it contains changes of sign, from $+$ to $-$ or from $-$ to $+$; and as many false [negative] roots as the number of times two $+$ signs or two $-$ signs are found in succession."

This astonishing claim, which many contemporaries hardly believed, and sometimes misinterpreted[7]), was subsequently improved by the statement that *the number of sign variations of a real polynomial simply is an upper bound to the number of positive roots, the difference being an even number.*

A complete proof was given by Gauss only[8]) in 1828!

Descartes' Rule of Signs, as the previous statement is now called, gives precise information about the positive roots of a polynomial only in two cases: when there are no variations at all and therefore the polynomial has no positive real roots, and when there is a single variation; in the latter case the polynomial has precisely one positive real root.

A deep generalization of Descartes' Rule of Signs is given by the following theorem of Budan and Fourier[9]).

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[6]) [34, p. 160]

[7]) See the letter of Carcavi to Descartes ([19], vol. V, p. 374) and Descartes' answer (ibidem, p. 397).

[8]) See [11] and the review by one of the authors in *Mathematical Reviews* 94d:01017.

[9]) The priority of Budan or Fourier has been a matter of historical dispute for a long time. Fourier's point of view is exposed by Navier in the *Avertissement de l'éditeur* of [11]. From a modern point of view the controversy appears rather pointless. The mere content of the theorem given by the two authors is the same, but Fourier emphasizes its benefit to localize the possible real roots, avoiding the unnecessary calculations that a naive use of Lagrange's "équation au carré des différences" implies ([11, p. 28]). Budan, on the other hand, has an amazingly modern understanding of the relevance of reducing the *algorithm* (his own word) to translate a polynomial by $x = x + p$, where p is an integer, to simple additions [15, pp. 11–16].

THEOREM 2.3. *Consider an n-th degree real polynomial $f(x)$ and two real numbers p, q with $p < q$. Then the sequence*

$$f(p), f'(p), f''(p), \ldots, f^{(n)}(p) \tag{2.1}$$

cannot have fewer variations than the sequence

$$f(q), f'(q), f''(q), \ldots, f^{(n)}(q). \tag{2.2}$$

The number of real roots of the equation $f(x) = 0$ included in the interval (p, q) equals the difference between the number of variations of the two sequences (2.1) *and* (2.2) *decreased, if necessary, by an even number.*

The choice $p = 0$ and $q = \infty$ immediately yields Descartes' result. The previous theorem provides a better understanding of Descartes' Rule: the role of the single sequence of the coefficients of a polynomial, originally used by Descartes, appears as the result of a very particular situation. Indeed Descartes' Rule is stated in terms of the number of variations of the sequence

$$\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n \tag{2.3}$$

of the coefficients of

$$\alpha_0 x^n + \alpha_1 x^{n-1} + \alpha_2 x^{n-2} + \cdots + \alpha_n \tag{2.4}$$

as a consequence of the fact that the search for the positive roots corresponds to the particular choice of the interval $(0, \infty)$.

In fact, for $x = 0$ the Fourier sequence

$$f(x), f'(x), f''(x), \ldots, f^{(n)}(x)$$

reduces to

$$0! \cdot \alpha_n, \; 1! \cdot \alpha_{n-1}, \; 2! \cdot \alpha_{n-2}, \; \ldots, \; n! \cdot \alpha_0, \tag{2.5}$$

whose terms differ by a positive factor from the terms of the sequence (2.3). The sequences (2.3) and (2.5) clearly have the same number of variations. For $x = \infty$ the Fourier sequence has no variations. Its role disappears and Descartes' rule may be formulated in terms of a single sequence.

In 1829 Sturm announced the following theorem (proved only in 1835), which seemed to establish definitely [10]) the accidental choice of the sequence (2.3) to investigate the number of positive real roots of the polynomial (2.4).

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[10]) The fascinating story of Sturm's theorem as well as the impressive number of algebraic researches it originated is described in [33]. For the sake of simplicity, we state the theorem in the case of the fundamental sequence whose first terms are $f(x)$ and $f'(x)$. Actually Sturm formulated the theorem in the more general terms of what was later called a "Sturm sequence" [7, pp. 341–349].

THEOREM 2.4. *Let $f(x)$ be an n-th degree real polynomial without multiple roots and consider the sequence of polynomials defined recursively by*

$$f_0(x) = f(x), \quad f_1(x) = \frac{df(x)}{dx},$$
$$f_k(x) = q_{k+1}(x) f_{k+1}(x) - f_{k+2}(x), \quad \textit{for} \quad k = 2, \dots, n-2,$$

where $q_{k+1}(x)$ is the quotient of $f_k(x)$ by $f_{k+1}(x)$ and $f_{k+2}(x)$ is the opposite of the remainder polynomial.

Then the number of zeros of $f(x)$ between p and q ($p < q$) equals the number of variations lost by the sequence

$$f_0(x), f_1(x), \dots, f_n(x)$$

when $x = p$ is replaced by $x = q$.

Sturm's theorem gives such a clear answer to the problem of determining the number of roots in a given interval that its algorithmic complexity was not considered relevant until the appearence of computer algebra. Let us see how it works through an example.

EXAMPLE 2.5. We take an example from [35]. Given the polynomial

$$f(x) = x^3 + 3x^2 - 4x + 1,$$

we want to know the number of its positive roots. Since $f'(x) = f_1(x) = 3x^2 + 6x - 4$, and we have

$$x^3 + 3x^2 - 4x + 1 = \frac{1}{3}(x+1)(3x^2 + 6x - 4) - \frac{7}{3}(2x - 1),$$

we deduce that

$$f_2(x) = \frac{7}{3}(2x - 1).$$

Again

$$f_1(x) = 3x^2 + 6x - 4 = \frac{9}{14}(x + \frac{5}{2}) \cdot \frac{7}{3}(2x - 1) - \frac{1}{4}$$

and so

$$f_3(x) = \frac{1}{4}.$$

Sturm's sequence is given by

$$\{x^3 + 3x^2 - 4x + 1,\ 3x^2 + 6x - 4,\ \frac{7}{3}(2x - 1),\ \frac{1}{4}\}.$$

For $x = 0$ the sequence becomes

$$\{1, -4, -\frac{7}{3}, \frac{1}{4}\}$$

and it has two variations. The limits as $x \to +\infty$ give the sequence $\{+\infty, +\infty, +\infty, +\infty\}$, which has no variations. We conclude that $f(x)$ has two positive roots. $\square$

REMARK 1. It is quite evident that Sturm's theorem also makes it possible to *isolate* the roots, i.e. to find disjoint intervals each containing a single root. Consider the previous example. If we evaluate Sturm's sequence at $x = 1$ we have

$$\{1, 5, \frac{7}{3}, \frac{1}{4}\}.$$

Since this sequence has no variations, the number of variations lost in passing from 0 to 1 is two, and it follows that the positive roots are located in $(0, 1)$. Let us evaluate the sequence for $x = \frac{1}{2}$ following an obvious bisection method. We have

$$\{-\frac{1}{8}, -\frac{1}{4}, 0, \frac{1}{4}\}.$$

It follows that Sturm's sequence loses one variation in passing from 0 to $\frac{1}{2}$ and loses one more variation in passing from $\frac{1}{2}$ to 1. Hence one root is located in $(0, \frac{1}{2})$ and the other in $(\frac{1}{2}, 1)$.

Considering the complete answer given by Sturm's theorem, the number of variations of a polynomial seems to be very weakly connected to the number of its positive roots, and the 'lucky' case given by 0 or 1 variations looks like an accident.

However we shall see that this situation may be considered the general one. Every polynomial has some sort of 'canonical forms' in which it assumes 0 or 1 variations. Moreover, these canonical forms can be obtained through an algorithm considerably less onerous than the one needed to implement Sturm's theorem.

In the sequel Δ denotes the '*least roots distance*' of the polynomial $f(x)$, that is the minimal distance

$$\min_{j<k} |\alpha_j - \alpha_k|$$

between distinct roots α_i of the equation $f(x) = 0$.