4. VINCENT'S PROOF OF HIS THEOREM

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A great merit of Vincent is to have understood perfectly the real aim of Lagrange. The requirement that a polynomial have a unique variation at a prescribed place is too demanding. We can be satisfied with the weaker requirement that a polynomial have a unique variation. This weakening gives the endpoints of the interval (a, b) a more balanced role. Moreover, in order to carry out a process for isolating the roots of an algebraic equation $f(x) = 0$, it is necessary to consider not only the behaviour of the polynomials f_h corresponding to the continued fraction expansions $[c_0, c_1, c_2, \dots]$ which approximate the roots, but also the other apparently purposeless expansions – and the related polynomials – which appear out of a systematic search for the roots [14]).

All this will be clarified by Example 5.2. To get to the point in question, let us give a precise statement.

THEOREM 4.1. *Consider an arbitrary real polynomial $f(x)$ of degree n, without multiple roots, and let $\gamma = [c_0, c_1, c_2, \dots]$, where the c_i are arbitrary positive integers for $i \geq 1$ and $c_0 \geq 0$, the k-th convergent being denoted by $\frac{p_k}{q_k}$. Define the sequence of variable substitutions*

$$x \leftarrow c_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \ddots + \cfrac{1}{c_h + \cfrac{1}{x}}}} = \frac{p_{h-1} + p_h x}{q_{h-1} + q_h x}, \qquad h = 0, 1, 2, \dots$$

Then, for h sufficiently large, the polynomial

$$f_{h+1}(x) = (q_{h-1} + q_h x)^n f\Big(\frac{p_{h-1} + p_h x}{q_{h-1} + q_h x}\Big)$$

has at most one variation.

Proof. To simplify the problem, we again follow Lagrange, setting $a_h = \frac{p_{h-1}}{q_{h-1}}$, $b_h = \frac{p_h}{q_h}$ and making the substitution $x \leftarrow \frac{q_{h-1}}{q_h} x$. We are reduced to studying the variations of the polynomial

$$\phi_{h+1}(x) = (1 + x)^n f\Big(\frac{a_h + b_h x}{1 + x}\Big). \tag{4.1}$$

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[14]) Via the Budan-Fourier theorem, for example.

For simplicity of notation, we hereafter denote a_h and b_h simply by a and b, and ϕ_{h+1} by ϕ.

Denote again by $x_0, x_1, \ldots, x_{n-1}$ the roots of $f(x)$, and by Δ the least distance between pairs of these roots.

The behaviour of real and complex roots is given by formulae (3.3) and (3.4). But Vincent makes a judicious observation: in order that the root ξ_k obtained from x_k via (3.4) have negative real part, it is enough to require that

$$(\rho_k - a)(b - \rho_k) - \sigma_k^2 < 0. \tag{4.2}$$

Considering (4.2) in geometrical terms [15]), we see that it is equivalent to asking that the point (ρ_k, σ_k) of the ρ–σ-plane should lie outside the circle whose equation is

$$\rho^2 + \sigma^2 - (a+b)\rho + ab = 0;$$

this circle is centered at $\left(\frac{a+b}{2}, 0\right)$ and its radius is $\frac{1}{2}|b-a|$.

But

$$\tfrac{1}{2}|b-a| = \frac{1}{2\, q_h\, q_{h-1}},$$

which shows that, as h increases, $\frac{1}{2}|b-a| \longrightarrow 0$. Condition (4.2) is then satisfied for h sufficiently large.

Assuming that h is large enough to satisfy (4.2) and the further inequality

$$|b-a| = \frac{1}{q_h\, q_{h-1}} < \Delta,$$

then at most one real root can belong to the interval (a, b).

Hence, for sufficiently large h, the polynomial (4.1) can be written as

$$K(x \pm \xi_0)(x+p) \cdot \ldots \cdot (x^2 + 2Rx + R^2 + S^2) \cdot \ldots,$$

where $p, \ldots, R, S$ are positive and we take the minus or plus sign in $(x \pm \xi_0)$ according to whether or not there exists a real root $x_0 \in (a, b)$ and $\xi_0 = \dfrac{x_0 - a}{b - x_0}$.

Let $g(x)$ be the polynomial whose transformed form under (3.2) is

$$G(x) = (x+p) \cdot \ldots \cdot (x^2 + 2Rx + R^2 + S^2) \cdot \ldots.$$

At this point Vincent observes that

$$\frac{a+bx}{1+x} = b + \frac{a-b}{1+x} = b + u.$$

[15]) Vincent actually uses a slightly different argument. He looks at the minimum value of the product $\left(\frac{p_h}{q_h} - \rho\right)\left(\frac{p_{h-1}}{q_{h-1}} - \rho\right)$.

Hence

$$(4.3)\qquad \begin{aligned} G(x) &= (1+x)^{n-1} g(b+u) \\ &= (1+x)^{n-1}\left[g(b) + g'(b)u + g''(b)\frac{u^2}{2!} + \cdots + g^{(n-1)}(b)\frac{u^{n-1}}{(n-1)!}\right]. \end{aligned}$$

Since $u \to 0$ as $h \to \infty$,

$$G(x) \to g(b)(1+x)^{n-1}$$

and the polynomial (4.1) has the limit

$$(4.4)\qquad K^*(x \pm \xi_0)(1+x)^{n-1}.$$

For h large enough, the number of variations of (4.1) is equal to the number of variations of (4.4). If we have the plus sign in the factor $x \pm \xi_0$, there are no variations.

Let us consider the case [16]) given by

$$(x - \xi_0)(1+x)^{n-1}.$$

We have

$$(1+x)^{n-1} = \sum_{k=0}^{n-1} a_k x^{n-1-k}, \quad \text{where} \quad a_k = \binom{n-1}{k},$$

and for $k = 1, 2, \dots,$

$$(4.5)\qquad a_k^2 - a_{k-1}a_{k+1} > 0.$$

Now

$$\begin{aligned} (x-\xi_0)(1+x)^{n-1} &= (x-\xi_0)\left[a_0 x^{n-1} + a_1 x^{n-2} + a_2 x^{n-3} + \dots\right] \\ &= x^n + [a_1 - a_0\xi_0]\,x^{n-1} + [a_2 - a_1\xi_0]\,x^{n-2} + \cdots - \xi_0. \end{aligned}$$

From (4.5) it is clear that, if for a given k the coefficient of x^{n-k},

$$a_k - a_{k-1}\xi_0,$$

is negative then all the subsequent coefficients are negative. Since the constant term is negative we have exactly one variation. □

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[16]) Obreschkoff's lemma, quoted in Section 6, immediately gives the result, but we want to follow Vincent's argument.

REMARK 4. We do not wish to deny Vincent's great value and originality, yet we find his proof disappointing. In fact, after a careful examination of the effect of the variable transformation $x \leftarrow \dfrac{a+bx}{1+x}$ to get information about the location of the roots of the polynomial $\phi(x)$, Vincent abruptly neglects what he has obtained and goes on to consider the effect of Taylor's formula applied to $(1+x)^{n-1}g(b+u)$. This approach carries no trace of all his previous work, and it is evident that the results one can obtain about the size of h are not best possible. A century later Uspensky modified the proof, but followed the same path, as we shall see later. Obviously we are not trying to criticize Vincent, but simply to emphasize the lack of consideration of the complex plane structure.

REMARK 5. While continued fractions appear naturally in the search for the roots of an algebraic equation, and are closely linked to the problem of separating the roots (see the following example), it is evident that they merely provide a tool, in the preceding proof, to get two sufficiently close values a, b. The theorem may be formulated entirely in terms of the transformation (3.2).

EXAMPLE 4.2. To see how Vincent's theorem can be used to separate the roots of an equation, we consider once again the polynomial of Example 2.5. The polynomial x^3+3x^2-4x+1 has two variations, hence the theorem of Budan and Fourier implies that the equation

$$x^3+3x^2-4x+1=0 \tag{4.6}$$

has either two or zero positive roots. By making the substitution $x \leftarrow 1+x$, we obtain the polynomial

$$(1+x)^3+3(1+x)^2-4(1+x)+1=x^3+6x^2+5x+1\,,$$

which has no variations and consequently has no positive roots. This shows that the equation (4.6) has no roots greater than 1. To consider the possibility of roots in $(0,1)$, we make the substitution $x \leftarrow \frac{1}{1+x}$. We obtain

$$(1+x)^3\left[\frac{1}{(1+x)^3}+3\frac{1}{(1+x)^2}-4\frac{1}{1+x}+1\right]=x^3-x^2-2x+1\,.$$

This polynomial still has two variations so it must again be subjected to the transformations $x \leftarrow 1+x$, $x \leftarrow \dfrac{1}{1+x}$. The transformed polynomials are

$$x^3+2x^2-x-1\,,\qquad x^3+x^2-2x-1\,.$$

Each of these has only one variation and hence has exactly one positive root. The first polynomial is obtained by the two successive substitutions

$$x \leftarrow \frac{1}{1+x}, \qquad x \leftarrow 1+x,$$

(or directly by $x \leftarrow \frac{1}{2+x}$). It follows that its positive root corresponds to a root of the original equation (4.6) in the interval $(0, \frac{1}{2})$. The second polynomial is obtained by the two identical substitutions

$$x \leftarrow \frac{1}{1+x}, \qquad x \leftarrow \frac{1}{1+x},$$

or

$$x \leftarrow \frac{1}{1+\dfrac{1}{1+x}}.$$

Its positive root corresponds to a root of (4.6) in $(\frac{1}{2}, 1)$.

REMARK 6. Looking at the previous example, we can be satisfied since the equation

$$x^3 + 3x^2 - 4x + 1 = 0$$

has one positive root in each of the intervals $(0, \frac{1}{2})$, $(\frac{1}{2}, 1)$. At this point, to approximate these roots we could use a suitable method such as Newton's or Dandelin's. But we can also proceed by using the same method. We know that the root of (4.6) in the interval $(\frac{1}{2}, 1)$ corresponds, via the substitution

$$x \leftarrow \frac{1}{1+\dfrac{1}{1+x}},$$

to the positive root of $x^3+x^2-2x-1 = 0$. Inserting the substitutions $x \leftarrow 1+x$, $x \leftarrow \dfrac{1}{1+x}$ into

$$x^3 + x^2 - 2x - 1 = 0$$

gives [17]

$$x^3 + 4x^2 + 3x - 1 = 0, \quad 1 + 6x + 5x^2 + x^3 = 0.$$

Since only the first equation has a variation, the root of the equation (4.6) is to be found by the transformation

$$x \leftarrow \frac{1}{1+\dfrac{1}{1+(1+x)}} = \frac{1}{1+\dfrac{1}{2+x}}.$$

[17]) The second susbstitution is useless, but we make it for the sake of clarity.

We find the new smaller interval $(\frac{2}{3}, 1)$. This remark explains the double role of Vincent's theorem, to isolate or to approximate the roots.

5. USPENSKY'S PROOF OF VINCENT'S THEOREM

Uspensky had the great merit of rediscovering Vincent's theorem and of providing the first modern proof. He also tried to popularize the use of the theorem as a powerful tool to isolate the roots of algebraic equations, but there he was unsuccessful, and it was only at the end of the seventies, mainly by the work of Akritas, that the root separation algorithm acquired its present status.

To clarify the structure of the proof, which at first sight looks rather cumbersome, we extract part of its content as an independent lemma, which is of little interest in itself, but will be used also in the proof of Section 6.

LEMMA 5.1. *If the n positive numbers*

$$R_k = \binom{n-1}{k}(1+\delta_k)\,, \quad k = 0, 1, \ldots, n-1\,,$$

are such that $|\delta_k| < \dfrac{1}{n}$, then the $n-1$ inequalities

$$R_k^2 - R_{k-1}R_{k+1} > 0\,, \quad k = 1, \ldots, n-1 \tag{5.1}$$

hold.

Proof. The inequalities (5.1) may be written as

$$\frac{(1+\delta_k)^2}{(1+\delta_{k-1})(1+\delta_{k+1})} > 1 - \frac{n}{(n-k)(k+1)}\,. \tag{5.2}$$

If $\varepsilon = \max\{|\delta_k|\}$, the left hand side of (5.2) is greater than

$$\frac{(1-\varepsilon)^2}{(1+\varepsilon)^2} = 1 - \frac{4\varepsilon}{(1+\varepsilon)^2}\,.$$

Hence (5.2) holds if

$$\frac{4\varepsilon}{(1+\varepsilon)^2} < \frac{n}{(n-k)(k+1)}\,. \tag{5.3}$$

The minimum value of

$$\frac{n}{(n-k)(k+1)}$$