2. The Serre spectral sequence

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **44 (1998)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **21.07.2024**

2. The Serre spectral sequence

The main theorem of this section relates the Euler characteristic of a Koszul complex on a module to the Samuel multiplicity of the module. Let A be a local ring, and let M be a finitely generated A-module of dimension at most k. Let $\mathfrak{a}$ be an ideal of A such that $M/\mathfrak{a}M$ has finite length. We recall that the associated Hilbert-Samuel polynomial $P_M^{\mathfrak{a}}(n)$ is defined to be the polynomial for which

$$P_M^{\mathfrak{a}}(n) = \text{length}(M/\mathfrak{a}^n M)$$

for large n. If the dimension of M is at most k, we define the Samuel multiplicity $e_k(\mathfrak{a}, M)$ to be $k!$ times the coefficient of n^k in $P_M^{\mathfrak{a}}(n)$ (if the dimension of M is less than k, $e_k(\mathfrak{a}, M)$ will be zero).

THEOREM 1. *With notation as above, let $x_1, \ldots, x_k$ be a sequence of elements of A, and let $\mathfrak{a}$ be the ideal generated by $x_1, \ldots, x_k$. Assume that $M/\mathfrak{a}M$ is a module of finite length. Let $K_\bullet$ be the Koszul complex on $x_1, \ldots, x_k$, and let*

$$\chi(K_\bullet \otimes M) = \sum_{i=0}^{k} (-1)^i \,\text{length}(H_i(K_\bullet \otimes M)).$$

Then

$$\chi(K_\bullet \otimes M) = e_k(\mathfrak{a}, M).$$

We sketch the argument used to prove this theorem. The main idea is to examine the spectral sequence defined by the filtration on $K_\bullet$ induced by powers of $\mathfrak{a}$. For each $n \geq 0$ and for each i we consider the quotient $\mathfrak{a}^n K_i / \mathfrak{a}^{n+1} K_i$. For each $r \geq 0$ we then take the subquotient $E_{i,n}^r$ of this module defined by

$$E_{i,n}^r = \frac{\{k_i \in \mathfrak{a}^n K_i \mid d_i(k_i) \in \mathfrak{a}^{n+r} K_{i-1}\} + \mathfrak{a}^{n+1} K_i}{(\{d_{i+1}(k_{i+1}) \mid k_{i+1} \in \mathfrak{a}^{n-r+1} K_{i+1}\} \cap \mathfrak{a}^n K_i) + \mathfrak{a}^{n+1} K_i}.$$

The $E_{i,n}^r$ define a spectral sequence (the usual spectral sequence associated to a filtered complex). While the precise definition is necessarily quite complicated, the idea is that $E_{i,n}^r$ is the subquotient of $\mathfrak{a}^n K_i / \mathfrak{a}^{n+1} K_i$ consisting of elements whose boundaries lie r steps further down in the filtration modulo boundaries of elements which lie at most $r-1$ steps further up in the filtration. As r gets large, this subquotient approaches the submodule of elements whose boundaries are zero modulo the submodule consisting of all of the boundaries.

In fact, it can be shown using the Artin-Rees lemma (see Serre [7]) that the spectral sequence does in fact converge to the $\mathfrak{a}$-adic filtration on the homology of $K_\bullet$.

Part of the general theory of spectral sequences, which can be verified directly in this case from the above definition, is that the boundary map d_i on $K_\bullet$ induces a map $d^r_{i,n}$ from $E^r_{i,n}$ to $E^r_{i-1,n+r}$ for each i, n and r, and that we have

$$E^{r+1}_{i,n} = \mathrm{Ker}(d^r_{i,n})/\,\mathrm{Im}(d^r_{i+1,n-r})\,.$$

Thus the modules at stage $r+1$ can be computed as the homology of those at the r^{th} stage under maps induced by the boundary maps of $K_\bullet$.

We next examine the complexes defined by $E^0_{i,n}$ and $E^1_{i,n}$.

If we let $r = 0$ in the above definition of $E^r_{i,n}$, the condition that $d_i(k_i) \in \mathfrak{a}^{n+r}K_{i-1}$ states that $d_i(k_i) \in \mathfrak{a}^n K_{i-1}$, which is always true since k_i is assumed to be in $\mathfrak{a}^n K_i$ and d_i is a module homomorphism. Similarly, the condition that $k_{i+1} \in \mathfrak{a}^{n+1}K_{i+1}$ implies that $d_{i+1}(k_{i+1}) \in \mathfrak{a}^{n+1}K_i$, so that when $r = 0$ the denominator in the above definition of $E^r_{i,n}$ is just $\mathfrak{a}^{n+1}K_i$. Hence $E^0_{i,n}$ is simply $\mathfrak{a}^n K_i/\mathfrak{a}^{n+1}K_i$. Furthermore, since $K_\bullet$ is the Koszul complex on the generators of $\mathfrak{a}$, the maps d_i are all zero modulo $\mathfrak{a}$, and the maps induced by the boundary maps d_i on $E^0_{i,n}$ are zero. It then follows that $E^1_{i,n}$ is also equal to $\mathfrak{a}^n K_i/\mathfrak{a}^{n+1}K_i$.

We next consider the maps $d^1_{i,n}$ induced by d_i on $E^1_{i,n}$; we denote this map $\bar{d}_i$. Since $K_\bullet$ is the Koszul complex on $x_1, \ldots, x_k$, the map d_i is defined by a matrix with $\pm x_i$ in certain positions and zeros in the remaining positions. Thus $\bar{d}_i$ is defined by the same matrix in which x_i is considered as a map from $\mathfrak{a}^n K_i/\mathfrak{a}^{n+1}K_i$ to $\mathfrak{a}^{n+1}K_{i-1}/\mathfrak{a}^{n+2}K_{i-1}$ for each n. Let $\overline{K}_i$ denote the associated graded module of K_i under the filtration by powers of $\mathfrak{a}$. Then $\bar{d}_i$ defines a map of degree one from $\overline{K}_i$ to $\overline{K}_{i-1}$, and the above description shows that the resulting complex is the Koszul complex on $\bar{x}_1, \ldots, \bar{x}_k$, where $\bar{x}_i$ denotes the image of x_i in the component of degree 1 of the associated graded ring $gr_{\mathfrak{a}}(A)$.

Thus we have shown that if for each i we let

$$\overline{K}_i = \bigoplus\nolimits_{n\geq 0} E^1_{i,n} = \bigoplus\nolimits_{n\geq 0} \mathfrak{a}^n K_i/\mathfrak{a}^{n+1}K_i\,,$$

the maps $d^1_{i,n}$ induced by d_i define a complex $\overline{K}_\bullet$ which is the Koszul complex on $\bar{x}_1, \ldots, \bar{x}_k$ over $gr_{\mathfrak{a}}(A)$.

Up to now we have considered the filtration on $K_\bullet$ without mentioning the module M. However, exactly the same argument holds for $K_\bullet \otimes M$, and we obtain a spectral sequence $E^r_{i,n}(M)$ which converges to the homology of

$K_\bullet \otimes M$ and such that the modules $E^1_{i,n}(M)$ form the Koszul complex induced by $\bar{x}_1, \ldots, \bar{x}_k$ on the associated graded module $gr_{\mathfrak{a}}(M)$. This Koszul complex can also be expressed as $\bar{K}_\bullet \otimes gr_{\mathfrak{a}}(M)$. Since we are assuming that $M/\mathfrak{a}M$ has finite length, the homology of the Koszul complex induced by $\bar{x}_1, \ldots, \bar{x}_k$ on $gr_{\mathfrak{a}}(M)$ also has finite length. Thus, since stage $r+1$ of the spectral sequence is obtained from the r^{th} stage by taking homology, the Euler characteristic is preserved from each stage of the spectral sequence to the next. Hence, since the spectral sequence converges to the homology of $K_\bullet \otimes M$, we have

$$\chi(K_\bullet \otimes_A M) = \chi\big(\bar{K}_\bullet \otimes_{gr_{\mathfrak{a}}(A)} gr_{\mathfrak{a}}(M)\big)\,.$$

To complete the proof that this Euler characteristic is equal to the Samuel multiplicity, we interpret the complex $\bar{K}_\bullet \otimes_{gr_{\mathfrak{a}}(A)} gr_{\mathfrak{a}}(M)$ as a complex of graded modules. Denote this complex $\bar{K}^M_\bullet$. Each module $\bar{K}^M_i$ has a Hilbert polynomial P_i such that

$$P_i(n) = \sum_{j=0}^{n-1} \text{length}(\bar{K}^M_i)_j\,,$$

where $(\bar{K}^M_i)_j$ denotes the component of $\bar{K}^M_i$ of degree j. However, since $\bar{K}^M_\bullet$ is a Koszul complex on the associated graded module of M, we also have

$$P_i(n) = \binom{k}{i} P^{\mathfrak{a}}_M(n-i)$$

for all i, where $P^{\mathfrak{a}}_M$ is the Hilbert polynomial of M. The shift by i in $\bar{K}^M_\bullet$ is necessary so that the boundary maps will be maps of graded modules of degree zero. By the additivity of Hilbert polynomials, $\sum_{i=0}^k (-1)^i P_i(n)$ gives the Hilbert polynomial defined by the homology of $\bar{K}^M_\bullet$, which is constant with value $\chi(\bar{K}^M_\bullet)$. But a direct computation (we prove a more general version of this in a later section) shows that $\sum_{i=0}^k (-1)^i \binom{k}{i} P^{\mathfrak{a}}_M(n-i)$ is $k!$ times the coefficient of n^k in $P^{\mathfrak{a}}_M(n)$, which proves the result.

The point of this computation is that it transforms questions about Euler characteristics into questions about Hilbert polynomials, which are often easier to deal with. We consider one particularly important case. Let R be a regular local ring, and let $\mathfrak{p}$ and $\mathfrak{q}$ be ideals of R such that $R/\mathfrak{p} \otimes R/\mathfrak{q}$ has finite length. Suppose $\mathfrak{q}$ is generated by a regular sequence $x_1, \ldots, x_k$. Then $\dim(R/\mathfrak{q}) = \dim(R) - k$, so that we have $\dim(R/\mathfrak{p}) \le \dim(R) - \dim(R/\mathfrak{q}) = k$, and $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim(R)$ if and only if $\dim(R/\mathfrak{p}) = k$. Since $x_1, \ldots, x_k$ is a regular sequence, the Koszul complex $K_\bullet$ on $x_1, \ldots, x_k$ is a free resolution of $R/\mathfrak{q}$. Thus $\mathrm{Tor}_i^R(R/\mathfrak{q}, R/\mathfrak{p})$ is the homology $H_i(K_\bullet \otimes R/\mathfrak{p})$. Applying the above theorem with $M = R/\mathfrak{p}$, we deduce that

$$\chi(R/\mathfrak{q}, R/\mathfrak{p}) = e_k(\mathfrak{q}, R/\mathfrak{p})\,.$$

Since the Samuel multiplicity $e_k(\mathfrak{q}, R/\mathfrak{p})$ is always non-negative and is positive if and only if the dimension of $R/\mathfrak{p}$ is equal to k, this proves the conjectures in this case.

Serre's proof of the multiplicity conjectures in the equicharacteristic case proceeded by reducing to the case of a regular sequence by reduction to the diagonal. If R is a power series ring $k[[X_1, \ldots, X_d]]$ and M and N are R-modules with $M \otimes_R N$ of finite length, he introduced a new set of variables $Y_1, \ldots, Y_d$ and considered N as a module over $k[[Y_1, \ldots, Y_d]]$. He then defined a "complete" tensor product $M \widehat{\otimes}_k N$ over k as a module over the ring $k[[X_1, \ldots, X_d, Y_1, \ldots, Y_d]]$ and showed that

$$\mathrm{Tor}_i^R(M, N) \cong \mathrm{Tor}_i^{k[[X_i, Y_j]]}(M \widehat{\otimes}_k N, k[[X_i, Y_j]]/(X_1 - Y_1, \ldots, X_d - Y_d))\,.$$

Since $X_1 - Y_1, \ldots, X_d - Y_d$ form a regular sequence, this proves the result for power series rings, and the conjectures for general equicharacteristic rings can be reduced to this case by completion and the Cohen structure theorems.

3. Gabber's reduction to regular embeddings

In this section we describe Gabber's use of de Jong's theorem on the existence of "regular alterations" to reduce the intersection conjectures to questions on regular embeddings in projective space over R.

As above, let R be a regular local ring and let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals of R such that $R/\mathfrak{p} \otimes R/\mathfrak{q}$ has finite length. Let d be the dimension of R, let r be the dimension of $R/\mathfrak{p}$ and let t be the dimension of $R/\mathfrak{q}$.

The following theorem of de Jong [2] makes the reduction to a question on regular embeddings possible:

THEOREM 2. *Let A be a local integral domain which is a localization of a ring of finite type over a discrete valuation ring. Then there exists a projective map $\phi\colon X \to \mathrm{Spec}(A)$ such that*

- *X is an integral regular scheme.*
- *If K is the quotient field of A, then the extension $k(X)$ of K is finite (we say that X is* generically finite *over* Spec(A)).