

## 4. SOME EXAMPLES

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## 4. SOME EXAMPLES

We give here three simple examples of the type of bigraded ring which might result from this construction. Each of these examples is obtained by taking a birational map from a regular scheme to  $\text{Spec}(R/\mathfrak{q})$ , and the last two are simple resolutions of singularities.

We first summarize the construction up to this point. We began with a regular local ring  $R$  and prime ideals  $\mathfrak{p}$  and  $\mathfrak{q}$ . We then took a regular subscheme  $Z'$  of  $\text{Proj}(R[X_0, \dots, X_n])$  which was generically finite over  $\text{Spec}(R/\mathfrak{q})$ . The next step was to replace  $R[X_0, \dots, X_n]$  with the associated graded ring of  $I$  tensored with  $R/\mathfrak{m} = k$ . The sheaf  $\mathcal{O}_{Y'}$  defined by  $B = R/\mathfrak{p}[X_0, \dots, X_n]$  was then replaced with the sheaf  $\mathcal{M}$  defined by  $gr_I(B)$ , again tensored with  $k$ . The assumption of regularity implies that  $I/I^2$  is locally free over  $A/I$ ; denote its rank  $r$ . Then the dimension of  $\mathcal{M}$  is at most  $r$ , and it is equal to  $r$  if and only if we had  $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim(R)$ . We note that the fiber  $Z'_s$  of  $Z'$  over the maximal ideal of  $R$  has dimension at most  $\dim(R/\mathfrak{q}) - 1$ , but apart from that we do not know much about it. It is the projective scheme defined by the graded ring  $(A/I) \otimes_R k$ , which is the part of degree zero in the grading in the second component of the bigraded ring we are considering.

For the first example, let  $R$  have dimension four, let  $t, u, v, w$  be a regular system of parameters, and define the prime ideals  $\mathfrak{p}$  and  $\mathfrak{q}$  by letting  $\mathfrak{p} = (t, u)$  and  $\mathfrak{q} = (v, w)$ . In this case,  $\text{Spec}(R/\mathfrak{q})$  is already regular, and we can simply take the projective scheme  $\text{Proj}(R[X]) = \text{Spec}(R)$ .

For a slightly more complicated example, consider the subscheme of the projective space  $\text{Proj}(R[X, Y])$  defined by the ideal  $I$  generated by  $v, w$ , and  $Xu - Yt$ . Then  $Z'$  is the blow up of  $R/\mathfrak{q}$  at the point defined by the maximal ideal, and the fiber over  $s$  is projective space of dimension 1. One could define similar examples in higher dimension.

For a third example, let  $R$  have dimension 2, and let  $I$  be generated by  $Xu - Yt, Zu - Xt, X^2 - YZ$ . The projective space  $P$  has dimension 2, and the fiber over the maximal ideal has codimension one in  $\text{Proj}(k[X, Y, Z])$  and thus has dimension one. The sheaf defined by  $I/I^2$  has rank 2, but  $I$  is minimally generated by three elements.

In the above examples it was not really necessary to reduce to projective space since the original quotients  $R/\mathfrak{q}$  were regular. We next consider an example where the original scheme is not regular. Let  $\mathfrak{m}$  be minimally generated by  $t, u$ , and let  $\mathfrak{q}$  be the principal ideal generated by  $t^2 - u^3$ .

We can resolve the singularity by letting  $Z'$  be defined by the ideal in  $R[X, Y]$  generated by  $t^2 - u^3$ ,  $uX - tY$ ,  $X^2 - uY^2$ . The fiber  $Z'_s$  in this case is  $\text{Proj}(k[X, Y]/(X^2))$ .

Finally, we consider the case where  $\mathfrak{q}$  is the determinantal ideal in  $R$  of dimension 4 generated by  $wu - t^2$ ,  $wv - tu$ , and  $tv - u^2$ . In this case the resolution can be found by taking the ideal  $I$  in  $R[X, Y, Z, W]$  generated by the following elements:

$$Z^2 - YW, YZ - XW, Y^2 - XZ, uW - vZ, uZ - vY, uY - vX, u^2 - tv, \\ tW - vY, tZ - vX, tY - uX, tu - wv, t^2 - wu, wW - vX, wZ - uX, wY - tX.$$

The fiber over the maximal ideal is a determinantal subvariety of dimension 1.

In a later section we will return to these examples and consider the question of computing the Euler characteristics  $\chi_{E_s}(\mathcal{O}_{Z'_s}, \mathcal{M})$  for sheaves  $\mathcal{M}$  defined as above by certain prime ideals  $\mathfrak{p}$  of  $R$ .

### 5. HILBERT POLYNOMIALS OF BIGRADED MODULES

In section 2 we showed how the Serre spectral sequence can be used to express the Euler characteristic defined by a Koszul complex in terms of the Samuel multiplicity. In this section we show that similar results hold in the present situation. We now let  $C$  denote the bigraded ring which we previously denoted  $gr_I(A) \otimes_R k$ , where  $C_{i,j}$  consists of the elements of  $(I^j/I^{j+1}) \otimes k$  of degree  $i$ . Thus in our present notation,  $E_s = \text{Proj}(C)$ , where the grading on  $C$  is that in the first coordinate. Let  $C_0$  denote the subring  $\bigoplus_i(C_{i,0})$ . Let  $r$  be the rank of  $I/I^2$ , and let  $M$  be a bigraded module defining a sheaf  $\mathcal{M}$  on  $E_s$  of dimension at most  $r$ ; we define the dimension of  $M$  to be the dimension of the associated sheaf. We consider the question of computing the Euler characteristic  $\chi_{E_s}(\mathcal{O}_{Z'_s}, \mathcal{M})$ , which we also denote  $\chi(C_0, M)$ .

Let

$$0 \rightarrow F_k \rightarrow \dots \rightarrow F_1 \rightarrow F_0 \rightarrow C_0 \rightarrow 0$$

be a complex of bigraded modules which defines a locally free resolution of  $C_0$  over  $C$ . For any finitely generated bigraded module  $N$ , we let  $P_N(m, n)$  be the Hilbert polynomial of  $N$ ; more precisely, we define  $P_N$  to be the polynomial in two variables such that

$$P_N(m, n) = \sum_{i=0}^{n-1} \text{length}(N_{m,i})$$