5. Hilbert polynomials of bigraded modules

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We can resolve the singularity by letting Z' be defined by the ideal in $R[X, Y]$ generated by $t^2 - u^3$, $uX - tY$, $X^2 - uY^2$. The fiber Z'_s in this case is $\mathrm{Proj}(k[X, Y]/(X^2))$.

Finally, we consider the case where $\mathfrak{q}$ is the determinantal ideal in R of dimension 4 generated by $wu - t^2$, $wv - tu$, and $tv - u^2$. In this case the resolution can be found by taking the ideal I in $R[X, Y, Z, W]$ generated by the following elements:

$$Z^2 - YW, YZ - XW, Y^2 - XZ, uW - vZ, uZ - vY, uY - vX, u^2 - tv,$$
$$tW - vY, tZ - vX, tY - uX, tu - wv, t^2 - wu, wW - vX, wZ - uX, wY - tX.$$

The fiber over the maximal ideal is a determinantal subvariety of dimension 1.

In a later section we will return to these examples and consider the question of computing the Euler characteristics $\chi_{E_s}(\mathcal{O}_{Z'_s}, \mathcal{M})$ for sheaves $\mathcal{M}$ defined as above by certain prime ideals $\mathfrak{p}$ of R.

5. Hilbert polynomials of bigraded modules

In section 2 we showed how the Serre spectral sequence can be used to express the Euler characteristic defined by a Koszul complex in terms of the Samuel multiplicity. In this section we show that similar results hold in the present situation. We now let C denote the bigraded ring which we previously denoted $gr_I(A) \otimes_R k$, where $C_{i,j}$ consists of the elements of $(I^j/I^{j+1}) \otimes k$ of degree i. Thus in our present notation, $E_s = \mathrm{Proj}(C)$, where the grading on C is that in the first coordinate. Let C_0 denote the subring $\bigoplus_i (C_{i,0})$. Let r be the rank of I/I^2, and let M be a bigraded module defining a sheaf $\mathcal{M}$ on E_s of dimension at most r; we define the dimension of M to be the dimension of the associated sheaf. We consider the question of computing the Euler characteristic $\chi_{E_s}(\mathcal{O}_{Z'_s}, \mathcal{M})$, which we also denote $\chi(C_0, M)$.

Let

$$0 \to F_k \to \cdots \to F_1 \to F_0 \to C_0 \to 0$$

be a complex of bigraded modules which defines a locally free resolution of C_0 over C. For any finitely generated bigraded module N, we let $P_N(m, n)$ by the Hilbert polynomial of N; more precisely, we define P_N to be the polynomial in two variables such that

$$P_N(m, n) = \sum_{i=0}^{n-1} \mathrm{length}(N_{m,i})$$

for large m and n. The degree of P_N is equal to the dimension of N (that is, the dimension of the sheaf defined by N on E_s). Let M be a bigraded module of dimension at most r as above. Then $M \otimes F_i$ has dimension at most r, and we have that the alternating sum of $P_{M\otimes F_i}$ is constant with value equal to $\chi(C_0, M)$.

We will prove this in a special case below (and reduce the non-negativity conjecture to this special case in the next section). We first briefly consider the question of constructing a resolution $F_\bullet$ of C_0. One method is to take the E^1 term of the Serre spectral sequence as defined in the previous section, starting from a locally free resolution of A/I over A. However, even though A/I has a nice resolution by sums of shifts of A, the resulting locally free sheaves in the resolution over the associated graded ring will not be so simple. An alternative approach is to take a global Kozsul complex

$$\cdots\ \Lambda^2(I/I^2) \otimes gr_I(A) \to \Lambda^1(I/I^2) \otimes gr_I(A) \to gr_I(A) \to A/I \to 0\,.$$

The resolution over C can then be obtained in either of these constructions by tensoring with k. This resolution gives an expression for the Euler characteristic in terms of the Chern classes of I/I^2, but again it is not easy to see how to use this information to compute Euler characteristics.

For the remainder of this section we assume that $I/I^2 \otimes_R k$ is a sum of copies of $\mathcal{O}_{Z'_s}(-k_i)$ for various k_i, so that C is a polynomial ring $C_0[T_1, \dots, T_r]$ over C_0, where T_i has degree $(k_i, 1)$ in the bigrading on C. As mentioned above, the non-negativity conjecture will be reduced to this situation in the next section. In this case the resolution is the usual Koszul complex on $T_1, \dots, T_r$, and the Hilbert polynomial of $M \otimes F_i$ is a sum of Hilbert polynomials of M with shifts in the degrees. Furthermore, the Koszul complex on $T_1, \dots, T_r$ is a tensor product of Koszul complexes on the individual T_i, and we can compute the Hilbert polynomial of the tensor product $K_\bullet(T_1, \dots, T_r) \otimes M$ by tensoring by each factor $K_\bullet(T_i)$ in turn and keeping track of the result. As above, assume that the dimension of M is at most r, and let $Q^r_M(m,n)$ be the component of $P_M(m,n)$ of degree r. Let T_i have degree $(k,1)$, and consider the Hilbert polynomial obtained by tensoring with the complex

$$0 \to C[(-k,-1)] \xrightarrow{T_i} C \to 0\,.$$

The Hilbert polynomial of the resulting complex $K_\bullet(T_i) \otimes M$ will be given by the polynomial whose value at (m,n) is $P_M(m,n) - P_M(m-k, n-1)$. We compute this difference for a monomial $m^i n^j$ and obtain

$$\begin{aligned} m^i n^j - (m-k)^i (n-1)^j &= m^i n^j - (m^i - ikm^{i-1} + \dots)(n^j - jn^{j-1} + \dots) \\ &= m^i n^j - m^i n^j + ikm^{i-1}n^j + jm^i n^{j-1} + \cdots = ikm^{i-1}n^j + jm^i n^{j-1} + \dots\,, \end{aligned}$$

where the remaining terms have lower degree. Since we are concerned with the component of highest degree, this suffices for our computation. We note that we can express this result by the formula

$$Q^{r-1}_{K_\bullet(T_i)\otimes M} = \frac{\partial Q^r}{\partial n} + k\frac{\partial Q^r}{\partial m}\,.$$

Iterating this process, where we let T_i have degree $(k_i, 1)$ for each i, we have

$$\chi(C_0, M) = \prod_{i=1}^{r}\left(\frac{\partial}{\partial n} + k_i\frac{\partial}{\partial m}\right) Q^r_M\,.$$

In this formula Q^r_M could be replaced with P_M.

THEOREM 3. *Let $C = C_0[T_1, \dots, T_r]$, where T_i has degree $(k_i, 1)$ as above, and let M be a bigraded C-module of dimension at most r.*

(i) *If* $\dim(M) < r$, *then* $\chi(C_0, M) = 0$.

(ii) *If $k_i \geq 0$ for all i, then $\chi(C_0, M) \geq 0$.*

(iii) *If $k_i = 0$ for all i, then $\chi(C_0, M) > 0$ if and only if the coefficient of n^r in P_M is non-zero.*

(iv) *If $k_i > 0$ for all i and* $\dim(M) = r$, *then* $\chi(C_0, M) > 0$.

Proof. If the dimension of M is less than r, its Hilbert polynomial has degree less that r, so the result of taking r partial derivatives is zero. Thus (i) holds.

We prove (ii) and (iv) by induction on r. By taking a filtration of M, we may assume that M is of the form $(C/\mathfrak{p})[(i,j)]$, where $\mathfrak{p}$ is a bigraded prime ideal of C and $[(i,j)]$ denotes a shift in degrees. Suppose some T_i is not in $\mathfrak{p}$. Then T_i is a non-zero divisor on M, and we can tensor with the Koszul complex on T_i, replacing M with M/T_iM and reducing r by one. Thus the result follows by induction. If all T_i are in $\mathfrak{p}$, then its Hilbert polynomial is constant with respect to n, so we have $Q^r(m,n) = \alpha m^r$ for some $\alpha \geq 0$. Hence the above formula states that

$$\chi(C_0, M) = k_1k_2\cdots k_r(r!)\alpha\,.$$

If all the k_i are greater than or equal to zero, we thus have $\chi(C_0, M) \geq 0$. If all the k_i are greater than zero and M has dimension r, then $\alpha > 0$ and $\chi(C_0, M) > 0$. This proves (ii) and (iv).

If all the k_i are zero, then $\chi(C_0, M)$ is simply the r^{th} derivative of P_M, so it is positive if and only if the coefficient of n^r is positive. On the other hand, this coefficient gives the length of the module $\bigoplus_{i=1}^{n-1} M_{m,i}$ for sufficiently large n up to terms of lower degree in n, so it cannot be negative.

The graded ring obtained from the original situation will be of the form considered here when I is globally defined by a regular sequence, and the k_i will then be the degrees of the generators. We give an example to show that the condition that M has dimension r does not suffice for $\chi(C_0, M)$ to be positive. Let R have dimension 3 and let t, u, v be a regular system of parameters. Let $\mathfrak{q}$ be the ideal generated by v, and let I be the ideal of $R[X, Y]$ generated by v and $uX - tY$. Then the fiber over the closed point is projective space of dimension one, $C_0 = k[X, Y]$, and $C = C_0[T_1, T_2]$ with $k_1 = 0$ and $k_2 = 1$. Then if $M = C/T_1$, M has dimension 2 and $\chi(C_0, M) = 0$.

EXERCISE. Prove (without using the Serre positivity conjecture) that the module M in the previous paragraph could not arise from a prime ideal $\mathfrak{p}$ such that $R/\mathfrak{p} \otimes R/\mathfrak{q}$ has finite length and $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim(R)$.

6. GABBER'S PROOF OF NON-NEGATIVITY

In this section we complete Gabber's proof of the non-negativity of intersection multiplicities. We have seen in the last section that if $gr_I(A) \otimes_R k$ is a polynomial ring over $(A/I) \otimes_R k$ generated by elements of non-negative degree, then non-negativity follows. We show here that we can embed $gr_I(A) \otimes_R k$ into a polynomial ring of this type. Let A_0 denote $A/I \otimes_R k$. Actually, we show instead that we can embed the symmetric algebra $\mathrm{Sym}_{A_0}((I/I^2) \otimes_R k)$ into a polynomial ring by a locally flat map. Since I/I^2 is locally free, the map from the symmetric algebra to the associated graded algebra defines an isomorphism of schemes, so this suffices to prove the result. Let $S = \mathrm{Sym}_{A_0}((I/I^2) \otimes_R k)$.

Let E_s denote $\mathrm{Proj}(gr_I(A) \otimes_R k) = \mathrm{Proj}(\mathrm{Sym}_{A_0}((I/I^2) \otimes_R k))$ as above. Let $W = \mathrm{Proj}(A_0[T_1, \ldots, T_{r'}])$ for T_i of degree $(k_i, 1)$ for some integer r'. Suppose that f is a map from S into the polynomial ring $A_0[T_1, \ldots, T_{r'}]$ such that the map ϕ induced by f from W to E_s is flat of relative dimension $r' - r$, where r is the rank of I/I^2. Then we have a commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\phi} & A_0[T_1, \ldots, T_{r'}] \\ & \searrow \quad \swarrow & \\ & A_0 & \end{array}$$