6. Gabber's proof of non-negativity

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The graded ring obtained from the original situation will be of the form considered here when I is globally defined by a regular sequence, and the k_i will then be the degrees of the generators. We give an example to show that the condition that M has dimension r does not suffice for $\chi(C_0, M)$ to be positive. Let R have dimension 3 and let t, u, v be a regular system of parameters. Let $\mathfrak{q}$ be the ideal generated by v, and let I be the ideal of $R[X, Y]$ generated by v and $uX - tY$. Then the fiber over the closed point is projective space of dimension one, $C_0 = k[X, Y]$, and $C = C_0[T_1, T_2]$ with $k_1 = 0$ and $k_2 = 1$. Then if $M = C/T_1$, M has dimension 2 and $\chi(C_0, M) = 0$.

EXERCISE. Prove (without using the Serre positivity conjecture) that the module M in the previous paragraph could not arise from a prime ideal $\mathfrak{p}$ such that $R/\mathfrak{p} \otimes R/\mathfrak{q}$ has finite length and $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim(R)$.

6. GABBER'S PROOF OF NON-NEGATIVITY

In this section we complete Gabber's proof of the non-negativity of intersection multiplicities. We have seen in the last section that if $gr_I(A) \otimes_R k$ is a polynomial ring over $(A/I) \otimes_R k$ generated by elements of non-negative degree, then non-negativity follows. We show here that we can embed $gr_I(A) \otimes_R k$ into a polynomial ring of this type. Let A_0 denote $A/I \otimes_R k$. Actually, we show instead that we can embed the symmetric algebra $\mathrm{Sym}_{A_0}((I/I^2) \otimes_R k)$ into a polynomial ring by a locally flat map. Since I/I^2 is locally free, the map from the symmetric algebra to the associated graded algebra defines an isomorphism of schemes, so this suffices to prove the result. Let $S = \mathrm{Sym}_{A_0}((I/I^2) \otimes_R k)$.

Let E_s denote $\mathrm{Proj}(gr_I(A) \otimes_R k) = \mathrm{Proj}(\mathrm{Sym}_{A_0}((I/I^2) \otimes_R k))$ as above. Let $W = \mathrm{Proj}(A_0[T_1, \ldots, T_{r'}])$ for T_i of degree $(k_i, 1)$ for some integer r'. Suppose that f is a map from S into the polynomial ring $A_0[T_1, \ldots, T_{r'}]$ such that the map ϕ induced by f from W to E_s is flat of relative dimension $r' - r$, where r is the rank of I/I^2. Then we have a commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\ \phi\ } & A_0[T_1, \ldots, T_{r'}] \\ & \searrow \quad \swarrow & \\ & A_0 & \end{array}$$

which induces a commutative diagram of schemes

$$\begin{array}{ccc} E_s & \xleftarrow{\ \phi\ } & W \\ & \nwarrow \quad \nearrow & \\ & Z_s' & \end{array}$$

Let $\mathcal{M}$ be a sheaf on E_s defined by a bigraded module M. Since we are assuming that ϕ is flat, we have an isomorphism

$$\mathcal{T}or_i^{\mathcal{O}_{E_s}}(\mathcal{O}_{Z_s'}, \mathcal{M}) \cong \mathcal{T}or_i^{\mathcal{O}_W}(\mathcal{O}_{Z_s'}, \phi^*(\mathcal{M}))$$

for all i. Thus we have an equality of Euler characteristics

$$\chi_{E_s}(\mathcal{O}_{Z_s'}, \mathcal{M}) = \chi_W(\mathcal{O}_{Z_s'}, \phi^*(\mathcal{M}))\,.$$

Thus if we can find such a map f such that all the k_i are non-negative, the conjecture will follow. We now show that such an embedding exists. Gabber's proof uses the fact that the dual of I/I^2 over s is generated by global sections; we define this map directly without dualizing. At this point we assume that R is ramified. Although this is an unusual assumption, it is possible to reduce to the ramified case by a finite flat extension of R, for example by adjoining a square root of p, where R has mixed characteristic p. Let $t_1, \ldots, t_d$ be a minimal set of generators of the maximal ideal $\mathfrak{m}$ of R. Since R is ramified, $R/\mathfrak{m}^2$ is isomorphic to a polynomial ring in $t_1, \ldots, t_d$ modulo the square of the ideal generated by $t_1, \ldots, t_d$. Thus for each i, the partial derivative $\frac{\partial}{\partial t_i}$ defines a map from $R/\mathfrak{m}^2[X_0, \ldots, X_n]$ to $R/\mathfrak{m}[X_0, \ldots, X_n]$. By taking the composition with the map from I to $R/\mathfrak{m}^2[X_0, \ldots, X_n]$ induced by the inclusion of I into A and with the map from $R/\mathfrak{m}[X_0, \ldots, X_n] = A \otimes_R k$ to $(A/I) \otimes_R k = A_0$, we obtain a map from I to A_0. Since for all a in A and i in A the partial derivatives satisfy

$$\frac{\partial(ai)}{\partial t_i} = a\frac{\partial i}{\partial t_i} + i\frac{\partial a}{\partial t_i}$$

and A_0 is annihilated by I, we can deduce that the map induced by $\frac{\partial}{\partial t_i}$ vanishes on I^2 and defines a homomorphism of A/I-modules from I/I^2 to A_0.

Similarly, for each $i = 0, \ldots, n$ we have a map induced by $\frac{\partial}{\partial X_i}$ from I/I^2 to $A_0[-1]$, where the shift in degree arises from the fact that these partial derivatives lower the degree by 1.

Putting these together, we have a map from I/I^2 to $A_0^d \oplus A_0[-1]^{n+1}$, which define a map f from S to $A_0[T_1, \ldots, T_d, S_0, \ldots, S_n]$ where the T_i have degree 0 and the S_i have degree 1.

THEOREM 4. *The map ϕ is locally an inclusion of polynomial rings. In particular, it is locally flat of relative dimension $d + n + 1 - r$.*

Proof. This is one of the main points of the proof, and it is the only place where the full strength of the assumption that Z' is regular is used. It suffices to show that for every closed point p of Z'_s, the map from I/I^2 to $A_0^d \oplus A_0[-1]^{n+1}$ defines a split inclusion locally at the point p. We assume that the residue field is algebraically closed (which we can do by a flat extension) and look at the maximal ideal $\mathfrak{m}_p$ corresponding to p. The local ring at p in P, which we denote A_p, is isomorphic to $R[u_1, \ldots, u_n]_{\mathfrak{m}_p}$, where, after a change of coordinates, we may assume that $u_1, \ldots, u_n$ together with a set of generators of $\mathfrak{m}_R$ generate $\mathfrak{m}_p$. Since Z' is regular, I is generated locally by part of a regular system of parameters $i_1, \ldots, i_r$. Furthermore, the quotient I/I^2 is locally generated by the images of $i_1, \ldots, i_r$. Since $i_1, \ldots, i_r$ form part of a regular system of parameters, the images of their partial derivatives in $(A_p/\mathfrak{m}_p)^d \oplus (A_p/\mathfrak{m}_p)^{n+1}$ are linearly independent. Hence the map from $(I/I^2) \otimes k$ to $A_0^d \oplus A_0[-1]^{n+1}$ locally defines a split inclusion at p as was to be shown.

This completes the proof of the Serre nonnegativity conjecture. Since certain of the indeterminates in the polynomial ring used in the proof have degree zero, it does not show that the Euler characteristics must be positive. In fact, as we showed at the end of the previous section, the locally free sheaf defined by I/I^2 is not itself positive enough to ensure positivity. Thus the positivity conjecture requires studying the sheaf $\mathcal{M}$ coming from the associated graded ring of I on $R/\mathfrak{p}[X_0, \ldots, X_n]$.

We note that we can embed $A[-1]$ into A^{n+1} by a locally split embedding which sends 1 to $(X_0, \ldots, X_n)$ and thus embed S into a polynomial ring D generated by $d + (n+1)^2$ elements all of which have degree zero. Thus one criterion for the positivity conjecture to hold is that if we take the quotient of D by the image in D of the kernel of the map from $gr_I(A)$ to $gr_I(R/\mathfrak{p}[X_0, \ldots, X_n])$, then (under the usual assumptions) the coefficient of $n^{d+(n+1)^2}$ in the Hilbert polynomial of this quotient is not zero.

We remark that the construction we have presented is quite computational in the sense that it is possible to compute the embedding ϕ explicitly in special cases. We give two simple examples. First, let R have dimension 2 with maximal ideal generated by t, u, let $A = R[X, Y]$, and let I be generated by $uX - tY$. Then $A_0 = k[X, Y]$. The map f to $A_0[S_1, S_2, T_0, T_1]$ induced by the partial derivatives sends $uX - tY$ to $-YS_1 + XS_2 + uT_0 - tT_1$, which, after dividing by $\mathfrak{m}$, is $-YS_1 + XS_2$. Let $\mathfrak{p} = (t, u)$. Then $uX - tY$ is zero modulo $\mathfrak{p}$, so the kernel on the map of graded rings is generated by the image of $uX - tY$ in I/I^2. Hence $\mathcal{M}$ is mapped to the sheaf associated to $A_0[S_1, S_2, T_0, T_1]/(-YS_1 + XS_2)$. It can be verified that this quotient satisfies the condition on Hilbert polynomials; the positivity condition also follows from the fact that $-YS_1 + XS_2$ has degree $(1, 1)$.

Finally, we consider the example from section 3 in which I is generated by $t^2 - u^3, uX - tY, X^2 - uY^2$. Then I/I^2 has rank 2. Taking derivatives, we see that the map ϕ (after dividing by $\mathfrak{m}$) satisfies $\phi(t^2 - u^3) = 0$, $\phi(uX - tY) = XS_1 - TS_2$, and $\phi(X^2 - uY^2) = -Y^2S_2 + 2XT_0$. To compute the result of intersecting with Y', where Y' is generated by an ideal $\mathfrak{p}$, it suffices to compute the kernel of the map from the symmetric algebra on I/I^2 to the associated graded ring of I on $R/\mathfrak{p}[X, Y]$ tensored with k, and then find the image of this kernel in $A_0[S_1, S_2, T_0, T_1]$. On the other hand, in this case $\mathrm{Proj}(A_0) = \mathrm{Proj}(k[X, Y]/(X^2))$ has dimension zero, so that the locally free sheaf defined by $(I/I^2) \otimes_R k$ is actually positive.

Similar examples can be computed from the other examples in section 3.

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