§1. Invariants of graphs

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§ 1. INVARIANTS OF GRAPHS

In this section we define an invariant of colored, oriented, trivalent, plane graphs.

Fix an integer $n \geq 2$ throughout this paper and put

$$\mathcal{N} = \{-(n-1)/2, -(n-1)/2+1, \ldots, (n-1)/2\}.$$

For disjoint subsets A_1 and A_2 of $\mathcal{N}$ we put

$$\pi(A_1, A_2) = \#\{(a_1, a_2) \in A_1 \times A_2 \mid a_1 > a_2\}.$$

Let G be an oriented, trivalent, plane graph with "color" or "flow" on each of its edges. Here a *flow* f is a map from the edge set to positive integers less than or equal to n such that for every vertex v in G the sum of its values on the edges coming into v is equal to that on the edges going out from v (see Figure 1.1). So we may say that G is a network with infinite capacity without source or sink. We also note that at each vertex two edges are "in" and one edge is "out", or two edges are "out" and one edge is "in". We call these two in- or out-edges the *legs* and one out- or in-edge the *head* of the vertex.

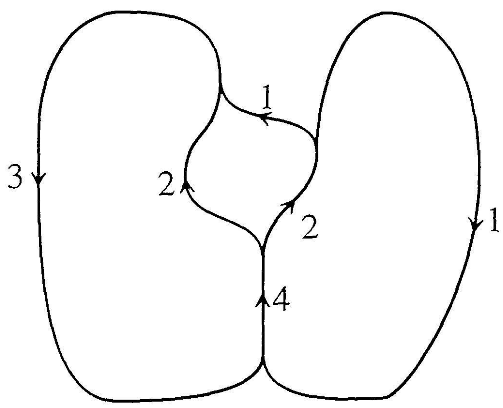

FIGURE 1.1
A graph with flow

A state σ is an assignment of a subset A of $\mathcal{N}$ to each edge e such that $\#(A) = f(e)$ and, moreover, at each vertex the union of subsets assigned to its legs coincides with that assigned to its head, where $\#(A)$ is the number of elements in A (see Figure 1.2). We denote by $\sigma(e)$ the subset of $\mathcal{N}$ assigned to an edge e.

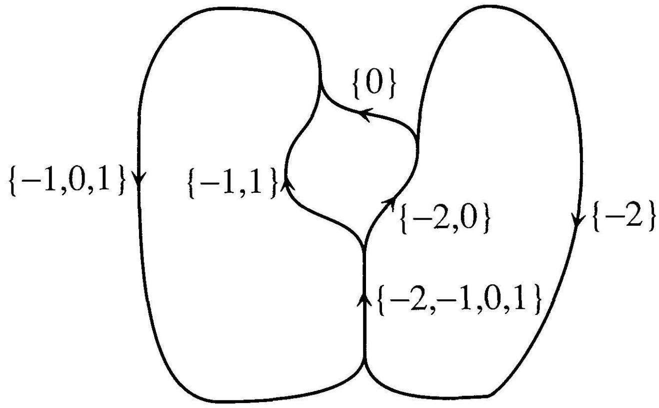

FIGURE 1.2
A state of the graph with flow in Figure 1.1

Given a state σ, we define the *weight* $\mathrm{wt}(v;\sigma)$ of a vertex v to be

$$q^{f(e_1)f(e_2)/4-\pi(\sigma(e_1),\sigma(e_2))/2},$$

where q is an indeterminate, and e_1 and e_2 are left and right legs respectively with respect to the orientation of G (Figure 1.3).

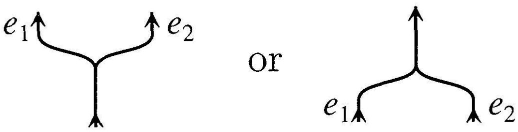

FIGURE 1.3

If we replace every edge e with $f(e)$ copies of parallel edges, assign each copy an element of the subset determined by σ, and connect at every vertex each pair of edges with the same element, we have a union of simple closed curves each of which equipped with the same element of $\mathcal{N}$ (Figure 1.4).

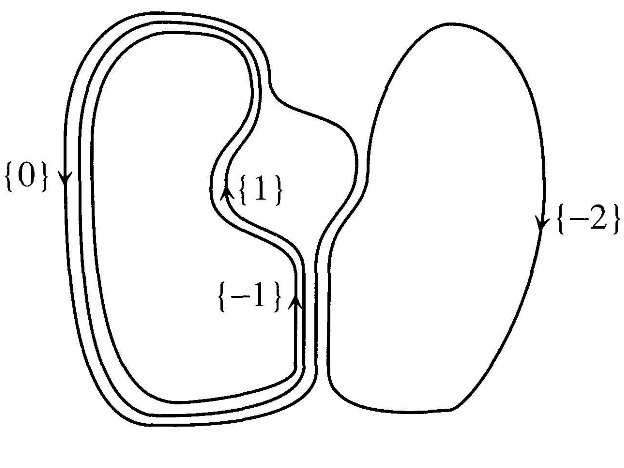

FIGURE 1.4
Simple closed curves defined by Figure 1.2

Then we define the *rotation number* $\mathrm{rot}(\sigma)$ to be

$$\sum_{C} \sigma(C)\,\mathrm{rot}(C)$$

where the sum is over all simple closed curves C equipped with $\sigma(C) \in \mathcal{N}$ and $\mathrm{rot}(C)$ is the rotation number of C (i.e., 1 if C is counter-clockwise and -1 otherwise). For example $\mathrm{rot}(\sigma) = 2$ for the state described in Figure 1.2 (see Figure 1.4).

Now we define $\langle G \rangle_n$ as follows.

$$\langle G \rangle_n = \sum_{\sigma:\text{state}} \Big\{ \prod_{v:\text{vertex}} \mathrm{wt}(v;\sigma) \Big\} q^{\mathrm{rot}(\sigma)} .$$

We define $\langle \text{empty graph} \rangle_n = 1$. It is clear that this is invariant under ambient isotopy of $\mathbf{R}^2$. Note that our invariant can be regarded as a colored graph invariant introduced by N. Yu. Reshetikhin and V. G. Turaev in [14] replacing each vertex by a "coupon". The coupon with two legs in would correspond to a projection $V_i \otimes V_j \to V_{i+j}$ and that with two legs out to an inclusion $V_{i+j} \to V_i \otimes V_j$, where V_i is the irreducible representation of $SU(n)$ corresponding to the i-fold anti-symmetric tensor of the vector representation.

§2. LOCAL PROPERTIES OF $\langle G \rangle_n$

We will describe some local properties of $\langle G \rangle_n$. In what follows diagrams indicated in each equality are identical outside the angle brackets $\langle \quad \rangle_n$ and each equality also holds if we reverse all the orientations of diagrams in both hand sides. A number near an edge indicates its flow. If a flow in a diagram exceeds n, we disregard the term where the diagram appears.

We put

$$[k] = \frac{q^{k/2} - q^{-k/2}}{q^{1/2} - q^{-1/2}} ,$$

$$[k]! = [1][2] \cdots [k] ,$$

and

$$\begin{bmatrix} i \\ j \end{bmatrix} = \frac{[i]!}{[j]![i-j]!} .$$

In the following equations we mean that if we replace the graph appearing in the left hand side with the one in the right hand side, we obtain the same value.