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**Autor:** Murakami, Hitoshi / Yamada, Shuji  
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§4. STATE MODEL OF TURAEV AND JONES

In this section we show that our definition of the HOMFLY polynomial gives the state model due to V. G. Turaev and V. F. R. Jones [16], [4]. Moreover we can recover the  $R$ -matrix for the  $q$ -deformation of the universal enveloping algebra  $U_q(\mathfrak{sl}(n, \mathbb{C}))$  found by M. Jimbo [2].

For readers' convenience, we first review Turaev's state model. Let  $R: V^{\otimes 2} \rightarrow V^{\otimes 2}$  be an isomorphism with  $V$  an  $n$ -dimensional vector space over  $\mathbb{C}$ . It is called an  $R$ -matrix if it satisfies the so-called Yang-Baxter equation:

$$(R \otimes \text{id}) \circ (\text{id} \otimes R) \circ (R \otimes \text{id}) = (\text{id} \otimes R) \circ (R \otimes \text{id}) \circ (\text{id} \otimes R).$$

Let  $\{e_1, e_2, \dots, e_n\}$  be a basis of  $V$  and  $R_{i,j}^{k,l}$  be the entry of  $R$  with respect to a fixed basis of  $V$ , which is the coefficient of  $e_k \otimes e_l$  of  $R(e_i \otimes e_j)$ .

Given a link diagram  $D$ , we regard it as a 4-valent graph (a crossing corresponding to a 4-valent vertex) and denote it by  $\tilde{D}$ . See Figure 4.1. Here the  $+$  ( $-$ , respectively) sign indicates that the vertex comes from a positive (negative, respectively) crossing.

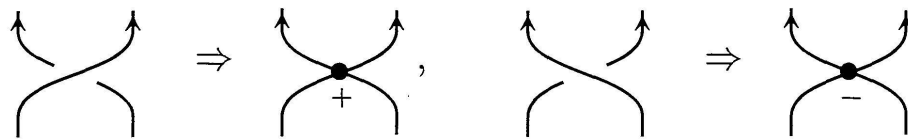


FIGURE 4.1  
Change crossings to 4-valent vertices

A state  $\tilde{\sigma}$  of  $\tilde{D}$  is a mapping from the edge set of  $\tilde{D}$  to  $\mathcal{N}$ . Then we define the weight  $\text{wt}_R(\tilde{v}, \tilde{\sigma})$  of a vertex  $\tilde{v}$  to be  $R_{ab}^{cd}$  if  $\tilde{v}$  comes from a positive crossing and  $(R^{-1})_{ab}^{cd}$  if it comes from a negative one, where  $a, b, c, d \in \mathcal{N}$  are the values of the four edges adjacent to  $\tilde{v}$  (see Figure 4.2).

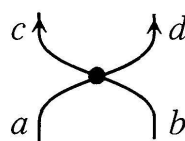


FIGURE 4.2  
State around a vertex  $\tilde{v}$

If we can define a suitable “rotation number”  $\widetilde{\text{rot}}(\tilde{\sigma})$  for each state, then the quantity

$$\sum_{\tilde{\sigma}:\text{state}} \left\{ \prod_{v:\text{vertex}} \text{wt}_R(\tilde{v}, \tilde{\sigma}) \right\} q^{\widetilde{\text{rot}}(\tilde{\sigma})}$$

becomes a framed link invariant.

Now we will proceed conversely and define an  $R$ -matrix from our framed link invariant  $\langle D \rangle_n$ .

First we define a weight  $\widetilde{\text{wt}}(\tilde{v}, \tilde{\sigma})$  by using  $\langle D \rangle_n$  as follows. We assume that for a state  $\tilde{\sigma}$ ,  $a, b, c, d \in \mathcal{N}$  appear around a vertex  $\tilde{v}$ . We define

$$\begin{aligned} \widetilde{\text{wt}} \left( \begin{array}{c} c \uparrow \\ \bullet \\ a \uparrow \quad b \uparrow \\ + \end{array} \right) &= \left\langle \begin{array}{c} \{c\} \uparrow \\ \{a\} \uparrow \quad \{b\} \uparrow \\ \end{array} \right\rangle_n \\ &= q^{1/2} \text{wt} \left( \begin{array}{c} \{c\} \uparrow \\ \{a\} \uparrow \end{array} \right) \left( \begin{array}{c} \{d\} \uparrow \\ \{b\} \uparrow \end{array} \right) - \text{wt} \left( \begin{array}{c} \{c\} \uparrow \quad \{d\} \uparrow \\ \{a\} \uparrow \quad \{b\} \uparrow \\ \{a,b\} \uparrow \end{array} \right) \end{aligned}$$

and

$$\begin{aligned} \widetilde{\text{wt}} \left( \begin{array}{c} c \uparrow \\ \bullet \\ a \uparrow \quad b \uparrow \\ - \end{array} \right) &= \left\langle \begin{array}{c} \{c\} \uparrow \\ \{a\} \uparrow \quad \{b\} \uparrow \\ \end{array} \right\rangle_n \\ &= q^{-1/2} \text{wt} \left( \begin{array}{c} \{c\} \uparrow \\ \{a\} \uparrow \end{array} \right) \left( \begin{array}{c} \{d\} \uparrow \\ \{b\} \uparrow \end{array} \right) - \text{wt} \left( \begin{array}{c} \{c\} \uparrow \quad \{d\} \uparrow \\ \{a\} \uparrow \quad \{b\} \uparrow \\ \{a,b\} \uparrow \end{array} \right). \end{aligned}$$

Here the first terms of the right hand sides are zero if  $a \neq c$  or  $b \neq d$  and the second terms are zero if  $\{a, b\} \neq \{c, d\}$ . Therefore, for  $a, b \in \mathcal{N}$  ( $a \neq b$ ) we have

$$\begin{aligned} \widetilde{\text{wt}} \left( \begin{array}{c} a \uparrow \\ \bullet \\ a \uparrow \quad b \uparrow \\ + \end{array} \right) &= q^{1/2} \text{wt} \left( \begin{array}{c} \{a\} \uparrow \\ \{a\} \uparrow \end{array} \right) \left( \begin{array}{c} \{b\} \uparrow \\ \{b\} \uparrow \end{array} \right) - \text{wt} \left( \begin{array}{c} \{a\} \uparrow \quad \{b\} \uparrow \\ \{a\} \uparrow \quad \{b\} \uparrow \\ \{a,b\} \uparrow \end{array} \right) \\ &= q^{1/2} - q^{\text{sign}(b-a)/2} \\ &= \begin{cases} q^{1/2} - q^{-1/2} & \text{if } a > b, \\ 0 & \text{if } a < b. \end{cases} \end{aligned}$$

Similarly we have

$$\widetilde{\text{wt}} \left( \begin{array}{c} a \uparrow \\ \bullet \\ b \uparrow \\ + \\ a \uparrow \\ b \uparrow \end{array} \right) = -1 \quad \text{and} \quad \widetilde{\text{wt}} \left( \begin{array}{c} a \uparrow \\ \bullet \\ a \uparrow \\ + \\ a \uparrow \\ a \uparrow \end{array} \right) = q^{1/2}.$$

We have a similar formula for a negative crossing.

Therefore we see that our graph invariant gives an  $R$ -matrix of the form

$$R_{ij}^{kl} = \begin{cases} q^{1/2} - q^{-1/2} & \text{if } i = k > j = l, \\ -1 & \text{if } i = l \neq j = k, \\ q^{1/2} & \text{if } i = j = k = l, \\ 0 & \text{otherwise;} \end{cases}$$

and

$$(R^{-1})_{ij}^{kl} = \begin{cases} -q^{1/2} + q^{-1/2} & \text{if } i = k < j = l, \\ -1 & \text{if } i = l \neq j = k, \\ q^{-1/2} & \text{if } i = j = k = l, \\ 0 & \text{otherwise;} \end{cases}$$

which coincides with  $-R(q^{1/2})^{-1}$ , where  $R(q^{1/2})$  is the  $R$ -matrix given in [16], replacing  $q$  with  $q^{1/2}$ .

## §5. INVARIANTS CORRESPONDING TO ANTI-SYMETRIC TENSORS

In this section, we will show briefly that our graph invariant gives the quantum link invariant each of its component equipped with an anti-symmetric tensor of the standard  $n$ -dimensional representation of  $SU(n)$ .

Let  $D$  be a link diagram each of its component colored with an integer  $i$  ( $1 \leq i \leq n$ ). This corresponds to the  $i$ -fold anti-symmetric tensor of the standard representation of  $SU(n)$ .