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Autor(en): **POPA, Sorin**

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ON CONNES' JOINT DISTRIBUTION TRICK AND A NOTION OF AMENABILITY FOR POSITIVE MAPS

by Sorin POPA [1])

0. INTRODUCTION

A key technical result in A. Connes' proof of the uniqueness of the injective type II_1 factor is a perturbation lemma, showing that if two positive, self-adjoint elements b_0, b_1 in a von Neumann algebra with a semifinite trace Tr are close to one another in the Hilbert norm given by Tr, $\|b_0 - b_1\|_{2,\mathrm{Tr}} < \varepsilon$, then most of their spectral projections are also close: $\|e_s(b_0) - e_s(b_1)\|_{2,\mathrm{Tr}} < f(\varepsilon)\|e_s(b_0)\|_{2,\mathrm{Tr}}$, with $f(\varepsilon) \to 0$ as $\varepsilon \to 0$. This result has since then become an indispensable tool in the analysis of type II_1 factors and semifinite von Neumann algebras in general. Actually, such estimates are known to be of basic importance in classical real analysis as well. But while elementary to prove for functions, they become quite non-trivial in the 'noncommutative framework' of the operators on the Hilbert space and were poorly dealt with before Connes' result.

The solution he gave to this is amazingly simple and ingenious, yet using only elementary functional analysis: since one would obviously like b_0, b_1 to commute, e.g., to be the coordinate functions on $\mathbf{R}^2$, then simply define a measure μ on the positive quadrant of $\mathbf{R}^2$ by requiring it to have the same joint distribution in the variables x, y as b_0, b_1 do with respect to Tr, i.e., $\mu([s, \infty) \times [t, \infty)) = \mathrm{Tr}(e_s(b_0)e_t(b_1))$. This perfectly determines μ and transfers the estimates in the Hilbert norm given by the trace, for b_0, b_1 and their spectral projections, into the same estimates for $H_0(x, y) = x$, $H_1(x, y) = y$ in $L^2(\mu)$, i.e, in a commutative setting!

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Note that the use of the L^2-norms in all this argument is imposed by the joint distribution trick and that, in fact, this requires a delicate handling of norm calculations, including the use of the Powers-Størmer inequality and of convexity properties of the Hilbert norm.

It is precisely the convexity of the Hilbert norm that we will further exploit in this paper so as to derive, by a slight adaptation of Connes' joint distribution trick and of the rest of his argument in [C], the following more general result:

THEOREM 0.1. *Let P_1, P_2 be semifinite von Neumann algebras with normal semifinite faithful traces, both of which are denoted by* Tr. *Let $\Phi_j : P_1 \to P_2$, $j = 0, 1, \ldots, n$, be positive, linear maps satisfying the conditions:*

(1) $\Phi_j(1) = 1$, $\mathrm{Tr} \circ \Phi_j \leq \mathrm{Tr}$, $j = 0, 1, \ldots, n$;

(2) $\sup\{ \|\Phi_j(x)\|_{2,\mathrm{Tr}} \mid x \in P_1, \ \|x\|_{2,\mathrm{Tr}} \leq 1 \} \leq 1$, $j = 0, 1, \ldots, n$.

Let $\delta > 0$ be such that $\delta < (5n)^{-32}$ and $b_0, b_1, \ldots, b_n \in P_{1+}$ satisfy the conditions:

(3) $\|b_j\|_{2,\mathrm{Tr}} = 1$, $\|\Phi_j(b_j)\|_{2,\mathrm{Tr}} \geq 1 - \delta$, $\forall j$;

(4) $\|\Phi_0(b_0) - \Phi_j(b_j)\|_{2,\mathrm{Tr}} < \delta$, $\forall j$.

Then there exists $s > 0$ such that

(i) $\|\Phi_0(e_s(b_0)) - \Phi_j(e_s(b_j))\|_{2,\mathrm{Tr}} < \delta^{1/4} \|e_s(b_0)\|$.

Moreover, if Φ_0 also satisfies $\mathrm{Tr} \circ \Phi_0 = \mathrm{Tr}$, *then there exists $s > 0$ such that, in addition to* (i), *we have*

(ii) $|\mathrm{Tr}(e_s(b_0)) - \mathrm{Tr}(e_s(b_j))| < \delta^{1/16} \mathrm{Tr}(e_s(b_0))$, $\forall j$

and

(iii) $\|\Phi_j(e_s(b_j))\|_{2,\mathrm{Tr}} > (1 - \delta^{1/32}) \|e_s(b_j)\|_{2,\mathrm{Tr}}$, $\forall j$.

Our interest in such a statement (which at first may seem a bit long and technical) comes from the following simple example: let $P_1 = P_2 = \ell^\infty(G)$, for G a discrete group, with Tr implemented by the counting measure on G. Take both $\Phi_0(f) = \Phi_1(f)$ to be the Markov operator $(1/n) \sum_{i=1}^{n} L_{g_i}(f)$, where $\{g_1, \ldots, g_n\}$ is a finite, self-adjoint set of elements of G, and L_{g_i} denotes the left translation operator by g_i. Note that Φ_0, Φ_1 satisfy (1), (2) and that if $\{g_1, \ldots, g_n\}$ contains the neutral element of G and generates G, then condition (3) for Φ_1 and all $\delta > 0$ amounts to Kesten's amenability condition for $(G; g_1, \ldots, g_n)$, requiring that the spectral radius of the Markov operator is equal to 1 ([K]). Assuming it is satisfied, let $\varepsilon > 0$ and $b \in \ell^2(G)$ be such that $\|b\|_2 = 1$, $\|\Phi_1(b)\|_2 \geq 1 - \delta$, where $\delta = (\varepsilon/n^2)^{32}$. Then $b_0 = b_1 = |b|$

clearly satisfy conditions (3) and (4). By part (iii) of the theorem, we thus have a spectral projection e of $b_0 = b_1$ such that $\|\Phi_1(e)\|_2 > (1 - \varepsilon/n^2)\|e\|_{2,\mathrm{Tr}}$. This clearly implies that if $F \subset G$ is the support set of e then F is finite and ε-invariant for $\{g_1, \dots, g_n\}$, thus showing that the group G satisfies Følner's amenability condition ([F], see also [Gr]).

So the above theorem can in fact be viewed as a general principle for positive maps between semifinite von Neumann algebras, leading from a "Kesten type condition" ((2) and (3) in our case) to a "Følner type condition" ((iii) in our case). To emphasize this more clearly we restate Theorem 0.1 for $n = 1$, $\Phi_0 = \Phi_1$, introducing along the lines a definition of amenability for positive maps between semifinite von Neumann algebras.

COROLLARY 0.2. *Let P_1, P_2 be semifinite von Neumann algebras with normal semifinite faithful traces denoted by* Tr. *Let $\Phi\colon P_1 \to P_2$ be a positive, linear, unital,* Tr*-preserving map. Then Φ satisfies*

$$\sup\{\|\Phi(x)\|_{2,\mathrm{Tr}} \mid x \in P_1,\ \|x\|_{2,\mathrm{Tr}} = 1\} = 1$$

if and only if it satisfies

$$\sup\{\frac{\|\Phi(e)\|_{2,\mathrm{Tr}}}{\|e\|_{2,\mathrm{Tr}}} \mid e \in \mathcal{P}(P_1), e \neq 0, \mathrm{Tr}(e) < \infty\} = 1\,.$$

DEFINITION 0.3. A positive, linear, unital, Trace-preserving map Φ between two semifinite von Neumann algebras P_1, P_2 is called *amenable* if it satisfies any of the equivalent conditions:

(a) *Kesten type condition:* $\sup\{\|\Phi(x)\|_{2,\mathrm{Tr}} \mid x \in P_1,\ \|x\|_{2,\mathrm{Tr}} = 1\} = 1$;

(b) *Følner type condition:* $\sup\{\frac{\|\Phi(e)\|_{2,\mathrm{Tr}}}{\|e\|_{2,\mathrm{Tr}}} \mid e \in \mathcal{P}(P_1),\ e \neq 0,\ \mathrm{Tr}(e) < \infty\} = 1$.

As an exemplification of this point of view, we will show in the last part of the paper how one can obtain a Følner type amenability condition for weighted bipartite graphs, from a Kesten type amenability condition on such graphs.

1. Proof of the theorem

Let $X = \mathbf{R}_+^2 \setminus \{0\}$ and $H_0(x,y) = x$, $H_i(x,y) = y$, $i = 1,2,\dots,n$. As in ([C], page 77), it follows that

$$\mu_i(A_0 \times A_i) \stackrel{\text{def}}{=} \mathrm{Tr}(\Phi_0(e_{A_0}(b_0))\Phi_i(e_{A_i}(b_i)))\,,$$

for $A_j \subset \mathbf{R}_+$, $0 \le j \le n$, Borel sets such that for each $i \ge 1$ either $0 \notin \bar{A}_0$ or $0 \notin \bar{A}_i$, defines a Radon measure μ_i on X, which satisfies the properties:

(a) $\|f(H_i)\|_{1,\mu_i} = \mathrm{Tr}(\Phi_i(|f|(b_i)))$ (resp., $\|f(H_i)\|_{2,\mu_i}^2 = \mathrm{Tr}(\Phi_i(|f|^2(b_i))) \le \|f(b_i)\|_{2,\mathrm{Tr}}^2$) for all Borel functions $f\colon [0,\infty) \to \mathbf{C}$ with $f(0) = 0$ and $f(b_i) \in L^1(P_1,\mathrm{Tr})$ (respectively $f(b_i) \in L^2(P_1,\mathrm{Tr})$), $i = 0,1,\dots,n$.

(b) $\displaystyle\int_X f_0(H_0)\overline{f_i(H_i)}\,\mathrm{d}\mu_i = \mathrm{Tr}(\Phi_0(f_0(b_0))\Phi_i(\bar{f}_i(b_i)))$, for all $f_i\colon [0,\infty) \to \mathbf{C}$ Borel with $f_i(0) = 0$ and $f_i(b_i) \in L^2(P_1,\mathrm{Tr})$, $\forall\, i = 0,1,\dots,n$.

(c) $\|f_0(H_0) - f_i(H_i)\|_{2,\mu} \ge \|\Phi_0(f_0(b_0)) - \Phi_i(f_i(b_i))\|_{2,\mathrm{Tr}}$, for all f_i as in (b).

(d) $\|H_0 - H_i\|_{2,\mu_i}^2 = \mathrm{Tr}(\Phi_0(b_0^2)) + \mathrm{Tr}(\Phi_i(b_i^2)) - 2\,\mathrm{Tr}(\Phi_0(b_0)\Phi_i(b_i)) \le 6\delta$.

Proof of (a)–(d). Indeed, (a) and (b) are clear by the proof of I.1 in [C] and the definition of μ_i. Further on, by (a), (b), (1), and Kadison's inequality (which asserts that positive, linear, unital maps φ between C^* algebras satisfy $\varphi(b)\varphi(b) \le \varphi(b^2)$ for any $b = b^*$), we get:

$$\begin{aligned}
\|f_0(H_0) - f_i(H_i)\|_{2,\mu_i}^2 &= \|f_0(H_0)\|_{2,\mu_i}^2 + \|f_i(H_i)\|_{2,\mu}^2 - 2\,\mathrm{Re}\int_X f_0(H_0)\overline{f_i(H_i)}\,\mathrm{d}\mu \\
&= \mathrm{Tr}(\Phi_0(f_0(b_0)^* f_0(b_0))) \\
&\qquad + \mathrm{Tr}(\Phi_i(f_i(b_i)^* f_i(b_i))) - 2\,\mathrm{Re}\,\mathrm{Tr}(\Phi_0(f_0(b_0))\Phi_i(\bar{f}_i(b_i))) \\
&\ge \mathrm{Tr}(\Phi_0(f_0(b_0))^*\Phi_0(f_0(b_0))) \\
&\qquad + \mathrm{Tr}(\Phi_i(f_i(b_i))^*\Phi_i(f_i(b_i))) - 2\,\mathrm{Re}\,\mathrm{Tr}(\Phi_0(f_0(b_0))\Phi_i(f_i(b_i))^*) \\
&= \|\Phi_0(f_0(b_0)) - \Phi_i(f_i(b_i))\|_{2,\mathrm{Tr}}^2\,.
\end{aligned}$$

This proves (c). Then (d) is clear by noticing that the hypothesis and the Cauchy-Schwarz inequality imply:

$$\begin{aligned}
\mathrm{Tr}(\Phi_0(b_0^2)) &+ \mathrm{Tr}(\Phi_i(b_i^2)) - 2\,\mathrm{Tr}(\Phi_0(b_0)\Phi_i(b_i)) \\
&\le \mathrm{Tr}(b_0^2) + \mathrm{Tr}(b_i^2) - 2\,\mathrm{Tr}(\Phi_0(b_0)\Phi_i(b_i)) \\
&= 2 - 2\,\mathrm{Tr}(\Phi_0(b_0)\Phi_i(b_i)) \\
&\le 2 - 2\,\mathrm{Tr}(\Phi_0(b_0)^2) + 2\delta \\
&\le 2(1 - (1-\delta)^2) + 2\delta \le 6\delta\,,
\end{aligned}$$

thus ending the proof of properties (a)–(d). $\square$

Proof of (i) *in the Theorem.* To prove (i), remark that we have, like in proof of 1.2.6 in [C], the estimate:

$$\int_{\mathbf{R}_+^*} \|e_{t^{1/2}}(H_0) - e_{t^{1/2}}(H_i)\|_{2,\mu_i}^2 \, dt$$
$$= \left\|H_0^2 - H_i^2\right\|_{1,\mu_i} \le \|H_0 - H_i\|_{2,\mu_i} \|H_0 + H_i\|_{2,\mu_i} .$$

But (d) implies $\|H_0 - H_i\|_{2,\mu_i} \le (6\delta)^{1/2}$ and (a) implies $\|H_0 + H_i\|_{2,\mu_i} \le \|H_0\|_{2,\mu_i} + \|H_i\|_{2,\mu_i} \le \|b_0\|_{2,\mathrm{Tr}} + \|b_i\|_{2,\mathrm{Tr}} = 2$. Thus, by applying (c) to the functions $f_i = \chi_{[t^{1/2},\infty)}$, $0 \le i \le n$, for each $t > 0$, and summing up the above inequalities over i we obtain

$$(*) \qquad \int_{\mathbf{R}_+^*} \sum_{i=1}^{n} \|\Phi_0(e_{t^{1/2}}(b_0)) - \Phi_i(e_{t^{1/2}}(b_i))\|_{2,\mathrm{Tr}}^2 \, dt$$
$$\le 2n(6\delta)^{1/2} = 2n(6\delta)^{1/2} \int_{\mathbf{R}_+^*} \|e_{t^{1/2}}(b_0)\|_{2,\mathrm{Tr}}^2 \, dt .$$

This implies that if we denote by D the set of all $t > 0$ for which

$$g(t) \overset{\text{def}}{=} \sum_{i=1}^{n} \|\Phi_0(e_{t^{1/2}}(b_0)) - \Phi_i(e_{t^{1/2}}(b_i))\|_{2,\mathrm{Tr}}^2 \, dt < \delta^{1/4} \|e_{t^{1/2}}(b_0)\|_{2,\mathrm{Tr}}^2 ,$$

then

$$\int_D \|e_{t^{1/2}}(b_0)\|_{2,\mathrm{Tr}}^2 \, dt \ge 1 - 5n\delta^{1/4} .$$

Indeed, from $\int_D \|e_{t^{1/2}}(b_0)\|_{2,\mathrm{Tr}}^2 \, dt < 1 - 5n\delta^{1/4}$, by taking into account that $g(t) \ge \delta^{1/4} \|e_{t^{1/2}}(b_0)\|_{2,\mathrm{Tr}}^2$ for $t \in \mathbf{R}_+^* \setminus D$, we would get:

$$\int_{\mathbf{R}_+^*} g(t) \, dt \ge \int_{\mathbf{R}_+^* \setminus D} g(t) \, dt$$
$$\ge \delta^{1/4} \int_{\mathbf{R}_+^* \setminus D} \|e_{t^{1/2}}(b_0)\|_{2,\mathrm{Tr}}^2 \, dt$$
$$\ge 5n\delta^{1/2} > 2n(6\delta)^{1/2} .$$

which is in contradiction with $(*)$.

In particular, since $\delta < (5n)^{-4}$, we have $1 - 5n\delta^{1/4} > 0$ so that $D \ne \varnothing$. Thus, any $s > 0$ with $s^2 \in D$ will satisfy (i).

Proof of (ii) *in the Theorem.* To prove (ii), note first that $\mathrm{Tr} \circ \Phi_0 \le \mathrm{Tr}$ already implies that for each fixed $x \in P_{2+}$ the map $L^1(P_1, \mathrm{Tr}) \ni x_1 \mapsto \mathrm{Tr}(x\Phi_0(x_1))$ defines a positive functional on $L^1(P_1, \mathrm{Tr})$, which we denote by $\Phi_0^*(x)$. Also, if we identify $L^1(P_1, \mathrm{Tr})^*$ with P_1, then $0 \leq x \leq 1$ implies $0 \leq \Phi_0^*(x) \leq 1$. Moreover, if in addition we have $\mathrm{Tr} \circ \Phi_0 = \mathrm{Tr}$, then $\Phi_0^*(1) = 1$, so Φ_0^* defines a positive, unital, linear mapping from P_2 into $P_1 = L^1(P_1, \mathrm{Tr})^*$ satisfying $\mathrm{Tr} \circ \Phi_0^* = \mathrm{Tr}$. Consequently, if we denote $\Phi_i' = \Phi_0^* \circ \Phi_i \colon P_1 \to P_1$, $1 \leq i \leq n$, then $\Phi_i'(1) = 1$, $\mathrm{Tr} \circ \Phi_i' \leq \mathrm{Tr}$, $\forall i$, $1 \leq i \leq n$, and we have the estimates:

$$\begin{aligned}\|\Phi_i'(b_i) - b_0\|_{2,\mathrm{Tr}}^2 &= \|\Phi_i'(b_i)\|_{2,\mathrm{Tr}}^2 + \|b_0\|_{2,\mathrm{Tr}}^2 - 2\,\mathrm{Tr}(\Phi_0^*(\Phi_i(b_i))b_0) \\ &\leq 2 - 2\,\mathrm{Tr}(\Phi_i(b_i)\Phi_0(b_0)) \\ &= 2 + \|\Phi_0(b_0) - \Phi_i(b_i)\|_{2,\mathrm{Tr}}^2 - \|\Phi_0(b_0)\|_{2,\mathrm{Tr}}^2 - \|\Phi_i(b_i)\|_{2,\mathrm{Tr}}^2 \\ &\leq 2 + \delta^2 - 2(1-\delta)^2 < 2\delta\,.\end{aligned}$$

In particular, if we denote $\delta' = (2\delta)^{1/2}$ then the above implies:

$$\|\Phi_i'(b_i)\|_{2,\mathrm{Tr}} \geq \|b_i\|_{2,\mathrm{Tr}} - (2\delta)^{1/2} = 1 - \delta'\,.$$

Altogether, this shows that we can apply the first part of the proof, with $\Phi_0' = \mathrm{id}, \Phi_1', \ldots, \Phi_n'$ instead of $\Phi_0, \Phi_1, \ldots, \Phi_n$ and δ' instead of δ, with the same $b_0, b_1, \ldots, b_n$, to obtain that the set D' of all $t > 0$ for which

$$\sum_{i=1}^{n} \|e_{t^{1/2}}(b_0) - \Phi_i'(e_{t^{1/2}}(b_i))\|_{2,\mathrm{Tr}}^2 < \delta'^{1/4} \|e_{t^{1/2}}(b_0)\|_{2,\mathrm{Tr}}^2$$

satisfies

$$\int_{D'} \|e_{t^{1/2}}(b_0)\|_{2,\mathrm{Tr}}^2 \,\mathrm{d}t \geq 1 - 5n\delta'^{1/4}\,.$$

Note that for $t \in D'$ we have:

$$\|e_{t^{1/2}}(b_0) - \Phi_i'(e_{t^{1/2}}(b_i))\|_{2,\mathrm{Tr}} < \delta'^{1/8} \|e_{t^{1/2}}(b_0)\|_{2,\mathrm{Tr}}$$

for all $i = 1, \ldots, n$. Due to this we also get:

$$\begin{aligned}\mathrm{Tr}(e_{t^{1/2}}(b_i)) &= \|e_{t^{1/2}}(b_i)\|_{2,\mathrm{Tr}}^2 \\ &\geq \|\Phi_i'(e_{t^{1/2}}(b_i))\|_{2,\mathrm{Tr}}^2 \geq (1 - \delta'^{1/8})^2 \|e_{t^{1/2}}(b_0)\|_{2,\mathrm{Tr}}^2 \\ &= (1 - \delta'^{1/8})^2\,\mathrm{Tr}(e_{t^{1/2}}(b_0)) > (1 - 2\delta'^{1/8})\,\mathrm{Tr}(e_{t^{1/2}}(b_0))\end{aligned}$$

for all $t \in D'$ and all $i = 1, \ldots, n$. Let then D_i' be the set of all $t \in D'$ for which $\mathrm{Tr}(e_{t^{1/2}}(b_i)) \leq (1 + \delta'^{1/16})\,\mathrm{Tr}(e_{t^{1/2}}(b_0))$. It follows that we have:

$$\begin{aligned}
\delta'^{1/16}\int_{D'\setminus D'_i}\mathrm{Tr}(e_{t^{1/2}}(b_0))\,\mathrm{d}t &\le \int_{D'\setminus D'_i}(\mathrm{Tr}(e_{t^{1/2}}(b_i))-\mathrm{Tr}(e_{t^{1/2}}(b_0)))\,\mathrm{d}t\\
&= \int_{D'}\mathrm{Tr}(e_{t^{1/2}}(b_i))-\mathrm{Tr}(e_{t^{1/2}}(b_0)))\,\mathrm{d}t\\
&\qquad + \int_{D'_i}(\mathrm{Tr}(e_{t^{1/2}}(b_0))-\mathrm{Tr}(e_{t^{1/2}}(b_i)))\,\mathrm{d}t\\
&\le \int_{\mathbf{R}^*_+\setminus D'}(\mathrm{Tr}(e_{t^{1/2}}(b_0))-\mathrm{Tr}(e_{t^{1/2}}(b_i)))\,\mathrm{d}t\\
&\qquad + 2\delta'^{1/8}\int_{D'_i}\mathrm{Tr}(e_{t^{1/2}}(b_0))\,\mathrm{d}t\\
&\le \int_{\mathbf{R}^*_+\setminus D'}\mathrm{Tr}(e_{t^{1/2}}(b_0))\,\mathrm{d}t + 2\delta'^{1/8}\\
&\le 5n\delta'^{1/4}+2\delta'^{1/8} < 3\delta'^{1/8},
\end{aligned}$$

in which we used, in the previous estimates, the identity

$$\int_{D'}(\mathrm{Tr}(e_{t^{1/2}}(b_i))-\mathrm{Tr}(e_{t^{1/2}}(b_0)))\,\mathrm{d}t = \int_{\mathbf{R}^*_+\setminus D'}(\mathrm{Tr}(e_{t^{1/2}}(b_0))-\mathrm{Tr}(e_{t^{1/2}}(b_i)))\,\mathrm{d}t$$

(which follows from the equalities

$$\|b_i\|^2_{2,\mathrm{Tr}} = \int_{\mathbf{R}^*_+}\mathrm{Tr}(e_{t^{1/2}}(b_i))\,\mathrm{d}t = \int_{\mathbf{R}^*_+}\mathrm{Tr}(e_{t^{1/2}}(b_0))\,\mathrm{d}t = \|b_0\|^2_{2,\mathrm{Tr}})$$

and the fact that $5n\delta'^{1/8} < 1$.

It thus follows that if we put $D'' = \bigcap_{i=1}^n D'_i \cap D$ and take into account that $\delta < (5n)^{-32}$, then we get:

$$\begin{aligned}
\int_{D''}\mathrm{Tr}(e_{t^{1/2}}(b_0))\,\mathrm{d}t &\ge \int_{D\cap D'}\mathrm{Tr}(e_{t^{1/2}}(b_0))\,\mathrm{d}t - \sum_{i=1}^n\int_{D'\setminus D'_i}\mathrm{Tr}(e_{t^{1/2}}(b_0))\,\mathrm{d}t\\
&\ge 1-5n\delta^{1/4}-5n\delta'^{1/4}-3n\delta'^{1/16}\\
&> 1-5n\delta^{1/32} > 0.
\end{aligned}$$

Thus $D'' \neq \varnothing$. But if $s > 0$ is such that $s^2 \in D''$ then from the above we have

$$|\mathrm{Tr}(e_s(b_0)) - \mathrm{Tr}(e_s(b_i))| < \delta^{1/16}\,\mathrm{Tr}(e_s(b_0)),$$

which ends the proof of (ii).

Proof of (iii) *in the Theorem.* Finally, (iii) follows now immediately from the last inequality above, since we have:

$$\begin{aligned}\|\Phi_i(e_s(b_i))\|_{2,\mathrm{Tr}} &\geq \|\Phi_0^*(\Phi_i(e_s(b_i)))\|_{2,\mathrm{Tr}} \\ &\geq \|e_s(b_0)\|_{2,\mathrm{Tr}} - \|e_s(b_0) - \Phi_i'(e_s(b_i))\|_{2,\mathrm{Tr}} \\ &> (1 - \delta'^{1/8})\|e_{t^{1/2}}(b_0)\|_{2,\mathrm{Tr}} \\ &> (1 - \delta'^{1/8})(1 + \delta^{1/16})^{-1/2}\|e_{t^{1/2}}(b_i)\|_{2,\mathrm{Tr}} \\ &\geq (1 - \delta^{1/32})\|e_{t^{1/2}}(b_i)\|_{2,\mathrm{Tr}} .\end{aligned}$$

This ends the proof of the last part of the theorem. □

Proof of Corollary 0.2. As for the Corollary in the Introduction, it follows readily from the Theorem, by taking $n = 1$, $\Phi_0 = \Phi_1 = \Phi$, once we observe that, since Φ is positive, it is selfadjoint, so

$$\begin{aligned}&\sup\left\{ \|\Phi(x)\|_{2,\mathrm{Tr}} \;\middle|\; x \in P_1,\ \|x\|_{2,\mathrm{Tr}} = 1\right\} \\ &\qquad = \sup\left\{ \|\Phi(x)\|_{2,\mathrm{Tr}} \;\middle|\; x \in P_1,\ x = x^*,\ \|x\|_{2,\mathrm{Tr}} = 1\right\},\end{aligned}$$

and also by noticing that if $x \in P_1$ is such that $x = x^*$ then $\|\Phi(|x|)\|_{2,\mathrm{Tr}} \geq \|\Phi(x)\|_{2,\mathrm{Tr}}$. Indeed, this is because by approximating x by step functions (through spectral calculus) we may assume $x = \sum_i c_i p_i$ for some real scalars c_i and finitely many, mutually orthogonal projections of finite trace p_i. Then, taking into account that $\Phi(p_i), \Phi(p_j) \geq 0$ implies $\mathrm{Tr}(\Phi(p_i)\Phi(p_j)) \geq 0$, we get:

$$\begin{aligned}\|\Phi(x)\|_{2,\mathrm{Tr}}^2 &= \textstyle\sum_{i,j} c_i \bar{c}_j \,\mathrm{Tr}(\Phi(p_i)\Phi(p_j)) \\ &\leq \textstyle\sum_{i,j} |c_i||\bar{c}_j| \,\mathrm{Tr}(\Phi(p_i)\Phi(p_j)) = \|\Phi(|x|)\|_{2,\mathrm{Tr}}^2 .\end{aligned}$$

□

2. APPLICATIONS

We shall apply Theorem 0.1 to a case when the semifinite algebras are in fact commutative. We mention that the noncommutativity will be implicitly present though, through the consideration of the positive maps. Note also that in the proof of the Corollary below, only part (i) in the conclusion of the Theorem is being used. In turn, the proof of this part of the Theorem is relatively short.

COROLLARY 2.1. *Let $T = (t_{kk'})_{k,k' \in K}$ be a symmetric matrix with non-negative entries, only finitely many of which are non-zero on each row and column and with $t_{kk'} \geq 1$ whenever different from 0. Assume that for some $\alpha > 0$ and $\delta > 0$ the following conditions are satisfied:*

(a) *There exists a positive (possibly unbounded) function $v \colon K \to \mathbf{R}_+^*$ such that $Tv = \alpha v$.*

(b) *If we denote $\|T\| = \sup\{\|Tb\|_2 \mid b \in \ell^2(K),\ \|b\|_2 = 1\}$, then $\alpha \geq \|T\| > (1 - \delta^2/2)\alpha$, in which we denoted by $\| \ \|_2$ the norm in $\ell^2(K)$.*

Then there exists a finite non-empty subset $F \subset K$ such that

$$\sum_{k \in \partial F} v_k^2 < (\alpha)^4 \delta^{1/4} \sum_{k \in F} v_k^2 ,$$

where $\partial F = \{k' \in K \setminus F \mid \exists k \in F$ with $t_{kk'} \neq 0\}$.

Before deriving 2.1 above from Theorem 0.1, let us point out right away a simple consequence of the hypothesis of 2.1, needed below, and which is in fact contained in the first 3 lines of the proof of 3.2 on page 281 of [Po3].

LEMMA 2.2. *Let $T = (t_{kk'})_{k,k' \in K}$ be a matrix with non-negative entries and only finitely many $t_{kk'} \neq 0$ on each row and column. Assume there exists $\alpha > 0$ and $v \colon K \to \mathbf{R}_+^*$ such that $Tv = \alpha v$. Then we have $\alpha v(k)/v(k') \geq t_{kk'}$, for all $k, k' \in K$.*

Proof. For each subset $S \subset K$ denote by $T(S) = \{k' \in K \mid \exists k \in S$, with $t_{kk'} \neq 0\}$. Also, if $w \colon K \to \mathbf{C}$ then w_S denotes its restriction to S. With these notations we have $\alpha v_S = (T v_{T(S)})_S$. Thus, if $k' \in T(S)$ and $k \in S$ is such that $t_{kk'} \neq 0$ then $\alpha v(k) \geq v(k') t_{kk'}$. If $t_{kk'} = 0$ there is nothing to prove. □

Proof of 2.1. Let $\lambda = \alpha^{-1}$ and $\Phi = \lambda V T V^{-1}$, where V is the diagonal matrix over K with entries $v(k) = v_k$, $k \in K$. Note that Φ defines a bounded positive linear operator from $P \stackrel{\text{def}}{=} \ell^\infty(K)$ into itself such that $\Phi(1) = 1$. Let Tr denote the trace on P given by the weights $(v_k^2)_{k \in K}$ on K, i.e., if $b \in P = \ell^\infty(K)$ then

$$\|b\|_{1,\mathrm{Tr}} = \sum_{k \in K} |b_k| v_k^2 .$$

For $a, b \colon K \to \mathbf{C}$, at least one of which has finite support, we denote $\langle a, b \rangle = \sum_{k \in K} a_k \overline{b}_k$. For each $b \in P = \ell^\infty(K)$ with finite support we then have:

$$\mathrm{Tr}(\Phi(b)) = \left\langle \Phi(b), V^2(1) \right\rangle = \left\langle b, \lambda V T V^{-1} V^2(1) \right\rangle$$
$$= \left\langle b, \lambda VTV(1) \right\rangle = \left\langle b, V^2(1) \right\rangle = \mathrm{Tr}(b)\,.$$

Thus $\mathrm{Tr} \circ \Phi = \mathrm{Tr}$. In particular, by Kadison's inequality, this implies $\|\Phi(a)\|_{2,\mathrm{Tr}} \leq \|a\|_{2,\mathrm{Tr}}$, $\forall a \in L^2(P, \mathrm{Tr})$.

Since $\|\lambda T\| > (1 - \delta^2/2)$, it follows that $\exists\ F_0 \subset K$ finite such that $T_0 =_{F_0} (\lambda T)_{F_0}$ satisfies $1 \geq \|T_0\| \geq 1 - \delta^2/2$. By the classical Perron-Frobenius theorem applied to T_0 (which is a finite symmetric matrix with nonnegative entries) it follows that there exists $b_0 \in \ell^\infty(K) \simeq P$, supported in the set F_0, with $b_0(k) \geq 0$, $\forall k$, and $\langle b_0, b_0 \rangle = 1$, such that $T_0 b_0 \geq (1 - \delta^2/2) b_0$. Thus, $\lambda T b_0 \geq (1 - \delta^2/2) b_0$.

Let then $b \overset{\mathrm{def}}{=} V^{-1}(b_0) \in \ell^\infty(K)$ and note that

$$\|b\|_{2,\mathrm{Tr}}^2 = \left\langle V^{-1}(b_0), V^2 V^{-1}(b_0) \right\rangle = \langle b_0, b_0 \rangle = 1\,.$$

Moreover, we have:

$$\begin{aligned} \|\Phi(b) - b\|_{2,\mathrm{Tr}}^2 &\leq 2 - 2\,\mathrm{Tr}(\Phi(b) b) \\ &= 2 - 2\left\langle \lambda V^{-1} T(b_0), V(b_0) \right\rangle \\ &= 2 - 2\langle \lambda T(b_0), b_0 \rangle \\ &\leq 2 - 2(1 - \delta^2/2) = 2\delta^2/2 = \delta^2\,. \end{aligned}$$

Thus $\|b - \Phi(b)\|_{2,\mathrm{Tr}} < \delta$ and $\|\Phi(b)\|_{2,\mathrm{Tr}} \geq 1 - \delta$, while $\|b\|_{2,\mathrm{Tr}} = 1$.

By Theorem 0.1 it follows that if $\delta < 5^{-32}$ then there exists a finite spectral projection e of b such that $\|\Phi(e) - e\|_{2,\mathrm{Tr}} < \delta^{1/4} \|e\|_{2,\mathrm{Tr}}$. Note that by approximating if necessary e in the norm $\|\ \ \|_{2,\mathrm{Tr}}$ by projections which are supported on finite subsets of K, we can obviously assume e itself is supported on a finite subset of K.

In particular we have:

$$\begin{aligned} \|(1-e)\Phi(e)\|_{2,\mathrm{Tr}}^2 &\leq \|(1-e)\Phi(e)\|_{2,\mathrm{Tr}}^2 + \|e - e\Phi(e)\|_{2,\mathrm{Tr}}^2 \\ &= \|e - \Phi(e)\|_{2,\mathrm{Tr}}^2 < \delta^{1/4} \|e\|_{2,\mathrm{Tr}}^2\,. \end{aligned}$$

Let $F \subset K$ be the support set of $e \in \ell^\infty(K) \simeq P$. By Lemma 2.2 we have $v_k^{-1} v_{k_0} \geq \lambda t_{kk_0}$ for all $k_0, k \in K$ for which $t_{kk_0} \neq 0$. Since $t_{kk_0} \geq 1$ for such k, k_0, we get $(\Phi)_{kk_0} = \lambda v_k^{-1} v_{k_0} t_{kk_0} \geq \lambda^2$, for all $k, k_0 \in K$ for which the entry (k, k_0) of Φ is nonzero. In particular, this shows that $\Phi(e)(1-e) \geq \lambda^2 \chi_{\partial F}$, where $\chi_{\partial F} \in \ell^\infty(K)$ is the characteristic function of $\partial F \subset K$. Thus we have

$$\begin{aligned}\lambda^4 \sum_{k\in\partial F} v_k^2 &= \|\lambda^2 \chi_F\|_{2,\mathrm{Tr}}^2 \\ &\leq \|(1-e)\Phi(e)\|_{2,\mathrm{Tr}}^2 < \delta^{1/4}\|e\|_{2,\mathrm{Tr}}^2 \\ &= \delta^{1/4}\sum_{k\in F} v_k^2\end{aligned}$$

giving in the end the estimate:

$$\sum_{k\in\partial F} v_k^2 < \alpha^4\delta^{1/4}\sum_{k\in F} v_k^2\,,$$

thus completing the proof. □

COROLLARY 2.3. *Let $\Gamma = (a_{kl})_{k\in K, l\in L}$ be a bipartite graph, with K and L labeling its even and respectively odd vertices and a_{kl} being the number of edges between the vertices k and l. Assume there exist $\alpha > 0$ and $\vec{v} = (v_k)_{k\in K}$, with $v_k > 0, \forall k \in K$ such that $\Gamma\Gamma^t\vec{v} = \alpha\vec{v}$. Then Γ satisfies the Kesten-type amenability condition $\|\Gamma\|^2 = \alpha$ if and only if it satisfies the Følner-type condition:*

$$\forall \varepsilon > 0,\ \exists F \subset K,\ \text{finite},\ F \neq \varnothing,\ \textit{such that} \sum_{k\in\partial F} v_k^2 < \varepsilon \sum_{k\in F} v_k^2\,.$$

Moreover, if this is the case, then Γ will satisfy the above Følner condition for any other weight vector $\vec{w} = (w_k)_k > 0$ with $\Gamma\Gamma^t\vec{w} = \alpha\vec{w}$.

Proof. Simply apply 2.1 to $T = \Gamma\Gamma^t$. Note that this statement can be easily derived from Corollary 0.2 in the introduction as well. □

Weighted bipartite graphs have become of particular interest in recent years due to their occurence in the Jones theory of subfactors of finite index ([J], [GHJ]). Thus, the consecutive inclusions of the higher relative commutants of a subfactor $N \subset M$ of finite index, $[M:N] < \infty$, are described by a pointed, bipartite graph $\Gamma_{N,M}$, called the *standard*, or *principal graph* of $N \subset M$. Moreover, $\Gamma_{N,M}$ has a canonical weight vector $\vec{v}$, given by the square roots of the local indices in the Jones tower, satisfying $\Gamma\Gamma^t\vec{v} = [M:N]\vec{v}$, when $N \subset M$ satisfies a certain extremality conditions, and $\Gamma\Gamma^t\vec{v} = [M:N]_{\min}\vec{v}$, in general, $[M:N]_{\min}$ being the minimal index for $N \subset M$ ([Hi]).

The amenability condition for such graphs, and more generally for arbitrary weighted bipartite graphs $\Gamma, \alpha, \vec{v}$, has been considered by the author in several papers and lectures starting in 1988, initially in the form of the Kesten type condition in 2.2 (see e.g., [Po1,2,4]). The Følner-type condition was first considered in [Po3] and the equivalence of the two conditions, for graphs of subfactors, was shown in [Po3,4] (see also [Po5] for an operatorial proof). Both these equivalent notions of amenability are important in the classification of subfactors ([Po1,2,3,5]). Thus, it has been proved that hyperfinite subfactors with amenable graph are completely classified by their higher relative commutants invariant (the *standard invariant*).

The above Corollary 2.3 shows that in fact the equivalence between the two notions of amenability holds true in a very general setting, for all bipartite graphs. This includes a more general class of graphs that appear in the theory of subfactors. To describe them, let us first note the following:

LEMMA 2.4. *Let $N \subset M$ be an extremal inclusion of type II_1 factors and assume $Q \subset N$ (respectively $M \subset P$) is a factor such that* $\dim(Q' \cap N) < \infty$ *(resp.* $\dim(M' \cap P) < \infty$*). Then the sequence of inclusions of finite dimensional algebras $Q' \cap N \subset Q' \cap M \subset Q' \cap M_1 \subset \dots$ (resp. $M' \cap P \subset N' \cap P \subset N_1' \cap P \subset \dots$), in which N_j, M_k give a Jones tunnel-tower for $N \subset M$, with their corresponding traces, are described by a bipartite graph Γ with a weight vector $\vec{t} = (t_k)_{k \in K}$ such that $\Gamma\Gamma^t \vec{t} = [M : N]\vec{t}$.*

Proof. The proof is identical to the proof of 1.7 in [Po6]. □

DEFINITION 2.5. Weighted bipartite graphs $\Gamma, \alpha, \vec{t}$ associated to an extremal subfactor $N \subset M$, with $\alpha = [M : N] < \infty$, and to a factor $Q \subset N$, with $\dim(Q' \cap N) < \infty$ (respectively $M \subset P$, with $\dim(M' \cap P) < \infty$), like in 2.4, are called *l-semi-standard graphs* (resp. *r-semi-standard graphs*). From 2.3 we can thus immediately infer:

COROLLARY 2.6. *A semi-standard graph Γ associated to a subfactor satisfies the Kesten-type condition $\|\Gamma\|^2 = [M : N]$ if and only if it satisfies the Følner-type condition:*

$$\forall \varepsilon > 0,\ \exists F \subset K,\ \text{finite},\ F \neq \emptyset,\ \textit{such that} \sum_{k \in \partial F} t_k^2 < \varepsilon \sum_{k \in F} t_k^2 .$$

Note added in proof. After this paper had been accepted for publication, we learned that A. Zuk had recently obtained a statement similar to the above Corollary 2.1, i.e., the equivalence between the Kesten and the Følner type amenability conditions for arbitrary weighted graphs (cf. Chapter 6 in

"Sur certaines propriétés spectrales du laplacien sur les graphes", University Paul Sabatier, Toulouse, thesis 1996). He proved this result by using different methods than ours. Note that Zuk's result generalized (unknowingly !) our previous similar statement which only covered the particular graphs coming from subfactors ([Po2,3,4]). On the other hand, our Corollary 0.2 in the present paper proves (by using Connes' distribution trick) an equivalence between Kesten and Følner type amenability conditions that is sensibly more general than all these prior results.

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Sorin Popa
Université de Genève
Section de Mathématiques
Case postale 240
CH-1211 Genève 24
Switzerland
e-mail: Sorin.Popa@math.unige.ch