0. Introduction

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ON CONNES' JOINT DISTRIBUTION TRICK AND A NOTION OF AMENABILITY FOR POSITIVE MAPS

by Sorin POPA [1])

0. INTRODUCTION

A key technical result in A. Connes' proof of the uniqueness of the injective type II_1 factor is a perturbation lemma, showing that if two positive, self-adjoint elements b_0, b_1 in a von Neumann algebra with a semifinite trace Tr are close to one another in the Hilbert norm given by Tr, $\|b_0 - b_1\|_{2,\mathrm{Tr}} < \varepsilon$, then most of their spectral projections are also close: $\|e_s(b_0) - e_s(b_1)\|_{2,\mathrm{Tr}} < f(\varepsilon)\|e_s(b_0)\|_{2,\mathrm{Tr}}$, with $f(\varepsilon) \to 0$ as $\varepsilon \to 0$. This result has since then become an indispensable tool in the analysis of type II_1 factors and semifinite von Neumann algebras in general. Actually, such estimates are known to be of basic importance in classical real analysis as well. But while elementary to prove for functions, they become quite non-trivial in the 'noncommutative framework' of the operators on the Hilbert space and were poorly dealt with before Connes' result.

The solution he gave to this is amazingly simple and ingenious, yet using only elementary functional analysis: since one would obviously like b_0, b_1 to commute, e.g., to be the coordinate functions on $\mathbf{R}^2$, then simply define a measure μ on the positive quadrant of $\mathbf{R}^2$ by requiring it to have the same joint distribution in the variables x, y as b_0, b_1 do with respect to Tr, i.e., $\mu([s, \infty) \times [t, \infty)) = \mathrm{Tr}(e_s(b_0)e_t(b_1))$. This perfectly determines μ and transfers the estimates in the Hilbert norm given by the trace, for b_0, b_1 and their spectral projections, into the same estimates for $H_0(x, y) = x$, $H_1(x, y) = y$ in $L^2(\mu)$, i.e, in a commutative setting!

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Note that the use of the L^2-norms in all this argument is imposed by the joint distribution trick and that, in fact, this requires a delicate handling of norm calculations, including the use of the Powers-Størmer inequality and of convexity properties of the Hilbert norm.

It is precisely the convexity of the Hilbert norm that we will further exploit in this paper so as to derive, by a slight adaptation of Connes' joint distribution trick and of the rest of his argument in [C], the following more general result:

THEOREM 0.1. *Let P_1, P_2 be semifinite von Neumann algebras with normal semifinite faithful traces, both of which are denoted by* Tr. *Let $\Phi_j\colon P_1 \to P_2$, $j = 0, 1, \dots, n$, be positive, linear maps satisfying the conditions:*

(1) $\Phi_j(1) = 1$, $\mathrm{Tr} \circ \Phi_j \leq \mathrm{Tr}$, $j = 0, 1, \dots, n$;

(2) $\sup\{ \|\Phi_j(x)\|_{2,\mathrm{Tr}} \mid x \in P_1,\ \|x\|_{2,\mathrm{Tr}} \leq 1 \} \leq 1$, $j = 0, 1, \dots, n$.

Let $\delta > 0$ be such that $\delta < (5n)^{-32}$ and $b_0, b_1, \dots, b_n \in P_{1+}$ satisfy the conditions:

(3) $\|b_j\|_{2,\mathrm{Tr}} = 1$, $\|\Phi_j(b_j)\|_{2,\mathrm{Tr}} \geq 1 - \delta$, $\forall j$;

(4) $\|\Phi_0(b_0) - \Phi_j(b_j)\|_{2,\mathrm{Tr}} < \delta$, $\forall j$.

Then there exists $s > 0$ such that

(i) $\|\Phi_0(e_s(b_0)) - \Phi_j(e_s(b_j))\|_{2,\mathrm{Tr}} < \delta^{1/4} \|e_s(b_0)\|$.

Moreover, if Φ_0 also satisfies $\mathrm{Tr} \circ \Phi_0 = \mathrm{Tr}$, *then there exists $s > 0$ such that, in addition to* (i), *we have*

(ii) $|\mathrm{Tr}(e_s(b_0)) - \mathrm{Tr}(e_s(b_j))| < \delta^{1/16}\, \mathrm{Tr}(e_s(b_0))$, $\forall j$

and

(iii) $\|\Phi_j(e_s(b_j))\|_{2,\mathrm{Tr}} > (1 - \delta^{1/32}) \|e_s(b_j)\|_{2,\mathrm{Tr}}$, $\forall j$.

Our interest in such a statement (which at first may seem a bit long and technical) comes from the following simple example: let $P_1 = P_2 = \ell^\infty(G)$, for G a discrete group, with Tr implemented by the counting measure on G. Take both $\Phi_0(f) = \Phi_1(f)$ to be the Markov operator $(1/n)\sum_{i=1}^n L_{g_i}(f)$, where $\{g_1, \dots, g_n\}$ is a finite, self-adjoint set of elements of G, and L_{g_i} denotes the left translation operator by g_i. Note that Φ_0, Φ_1 satisfy (1), (2) and that if $\{g_1, \dots, g_n\}$ contains the neutral element of G and generates G, then condition (3) for Φ_1 and all $\delta > 0$ amounts to Kesten's amenability condition for $(G; g_1, \dots, g_n)$, requiring that the spectral radius of the Markov operator is equal to 1 ([K]). Assuming it is satisfied, let $\varepsilon > 0$ and $b \in \ell^2(G)$ be such that $\|b\|_2 = 1$, $\|\Phi_1(b)\|_2 \geq 1 - \delta$, where $\delta = (\varepsilon/n^2)^{32}$. Then $b_0 = b_1 = |b|$

clearly satisfy conditions (3) and (4). By part (iii) of the theorem, we thus have a spectral projection e of $b_0 = b_1$ such that $\|\Phi_1(e)\|_2 > (1 - \varepsilon/n^2)\|e\|_{2,\mathrm{Tr}}$. This clearly implies that if $F \subset G$ is the support set of e then F is finite and ε-invariant for $\{g_1, \ldots, g_n\}$, thus showing that the group G satisfies Følner's amenability condition ([F], see also [Gr]).

So the above theorem can in fact be viewed as a general principle for positive maps between semifinite von Neumann algebras, leading from a "Kesten type condition" ((2) and (3) in our case) to a "Følner type condition" ((iii) in our case). To emphasize this more clearly we restate Theorem 0.1 for $n = 1$, $\Phi_0 = \Phi_1$, introducing along the lines a definition of amenability for positive maps between semifinite von Neumann algebras.

COROLLARY 0.2. *Let P_1, P_2 be semifinite von Neumann algebras with normal semifinite faithful traces denoted by* Tr. *Let $\Phi\colon P_1 \to P_2$ be a positive, linear, unital,* Tr*-preserving map. Then Φ satisfies*

$$\sup\{ \|\Phi(x)\|_{2,\mathrm{Tr}} \mid x \in P_1,\ \|x\|_{2,\mathrm{Tr}} = 1\} = 1$$

if and only if it satisfies

$$\sup\{ \frac{\|\Phi(e)\|_{2,\mathrm{Tr}}}{\|e\|_{2,\mathrm{Tr}}} \mid e \in \mathcal{P}(P_1), e \neq 0, \mathrm{Tr}(e) < \infty\} = 1 .$$

DEFINITION 0.3. A positive, linear, unital, Trace-preserving map Φ between two semifinite von Neumann algebras P_1, P_2 is called *amenable* if it satisfies any of the equivalent conditions:

(a) *Kesten type condition:* $\sup\{ \|\Phi(x)\|_{2,\mathrm{Tr}} \mid x \in P_1,\ \|x\|_{2,\mathrm{Tr}} = 1\} = 1$;

(b) *Følner type condition:* $\sup\{ \frac{\|\Phi(e)\|_{2,\mathrm{Tr}}}{\|e\|_{2,\mathrm{Tr}}} \mid e \in \mathcal{P}(P_1),\ e \neq 0,\ \mathrm{Tr}(e) < \infty\} = 1$.

As an exemplification of this point of view, we will show in the last part of the paper how one can obtain a Følner type amenability condition for weighted bipartite graphs, from a Kesten type amenability condition on such graphs.