1. Proof of the theorem

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Let $X = \mathbf{R}_+^2 \setminus \{0\}$ and $H_0(x,y) = x$, $H_i(x,y) = y$, $i = 1,2,\ldots,n$. As in ([C], page 77), it follows that

$$\mu_i(A_0 \times A_i) \stackrel{\text{def}}{=} \operatorname{Tr}(\Phi_0(e_{A_0}(b_0))\Phi_i(e_{A_i}(b_i))),$$

for $A_j \subset \mathbf{R}_+$, $0 \leq j \leq n$, Borel sets such that for each $i \geq 1$ either $0 \notin \bar{A}_0$ or $0 \notin \bar{A}_i$, defines a Radon measure μ_i on X, which satisfies the properties:

(a) $\|f(H_i)\|_{1,\mu_i} = \operatorname{Tr}(\Phi_i(|f|(b_i)))$ (resp., $\|f(H_i)\|_{2,\mu_i}^2 = \operatorname{Tr}(\Phi_i(|f|^2(b_i))) \leq \|f(b_i)\|_{2,\mathrm{Tr}}^2$) for all Borel functions $f\colon [0,\infty) \to \mathbf{C}$ with $f(0) = 0$ and $f(b_i) \in L^1(P_1,\mathrm{Tr})$ (respectively $f(b_i) \in L^2(P_1,\mathrm{Tr})$), $i = 0,1,\ldots,n$.

(b) $\int_X f_0(H_0)\overline{f_i(H_i)}\,\mathrm{d}\mu_i = \operatorname{Tr}(\Phi_0(f_0(b_0))\Phi_i(\bar{f}_i(b_i)))$, for all $f_i\colon [0,\infty) \to \mathbf{C}$ Borel with $f_i(0) = 0$ and $f_i(b_i) \in L^2(P_1,\mathrm{Tr})$, $\forall\, i = 0,1,\ldots,n$.

(c) $\|f_0(H_0) - f_i(H_i)\|_{2,\mu} \geq \|\Phi_0(f_0(b_0)) - \Phi_i(f_i(b_i))\|_{2,\mathrm{Tr}}$, for all f_i as in (b).

(d) $\|H_0 - H_i\|_{2,\mu_i}^2 = \operatorname{Tr}(\Phi_0(b_0^2)) + \operatorname{Tr}(\Phi_i(b_i^2)) - 2\operatorname{Tr}(\Phi_0(b_0)\Phi_i(b_i)) \leq 6\delta$.

Proof of (a)–(d). Indeed, (a) and (b) are clear by the proof of I.1 in [C] and the definition of μ_i. Further on, by (a), (b), (1), and Kadison's inequality (which asserts that positive, linear, unital maps φ between C^* algebras satisfy $\varphi(b)\varphi(b) \leq \varphi(b^2)$ for any $b = b^*$), we get:

$$\begin{aligned}
\|f_0(H_0) - f_i(H_i)\|_{2,\mu_i}^2 &= \|f_0(H_0)\|_{2,\mu_i}^2 + \|f_i(H_i)\|_{2,\mu}^2 - 2\operatorname{Re}\int_X f_0(H_0)\overline{f_i(H_i)}\,\mathrm{d}\mu \\
&= \operatorname{Tr}(\Phi_0(f_0(b_0)^* f_0(b_0))) \\
&\qquad + \operatorname{Tr}(\Phi_i(f_i(b_i)^* f_i(b_i))) - 2\operatorname{Re}\operatorname{Tr}(\Phi_0(f_0(b_0))\Phi_i(\bar{f}_i(b_i))) \\
&\geq \operatorname{Tr}(\Phi_0(f_0(b_0))^*\Phi_0(f_0(b_0))) \\
&\qquad + \operatorname{Tr}(\Phi_i(f_i(b_i))^*\Phi_i(f_i(b_i))) - 2\operatorname{Re}\operatorname{Tr}(\Phi_0(f_0(b_0))\Phi_i(f_i(b_i))^*) \\
&= \|\Phi_0(f_0(b_0)) - \Phi_i(f_i(b_i))\|_{2,\mathrm{Tr}}^2 .
\end{aligned}$$

This proves (c). Then (d) is clear by noticing that the hypothesis and the Cauchy-Schwarz inequality imply:

$$\begin{aligned}
\operatorname{Tr}(\Phi_0(b_0^2)) &+ \operatorname{Tr}(\Phi_i(b_i^2)) - 2\operatorname{Tr}(\Phi_0(b_0)\Phi_i(b_i)) \\
&\leq \operatorname{Tr}(b_0^2) + \operatorname{Tr}(b_i^2) - 2\operatorname{Tr}(\Phi_0(b_0)\Phi_i(b_i)) \\
&= 2 - 2\operatorname{Tr}(\Phi_0(b_0)\Phi_i(b_i)) \\
&\leq 2 - 2\operatorname{Tr}(\Phi_0(b_0)^2) + 2\delta \\
&\leq 2(1 - (1-\delta)^2) + 2\delta \leq 6\delta,
\end{aligned}$$

thus ending the proof of properties (a)–(d). □

Proof of (i) *in the Theorem.* To prove (i), remark that we have, like in proof of 1.2.6 in [C], the estimate:

$$\int_{\mathbf{R}_+^*} \|e_{t^{1/2}}(H_0) - e_{t^{1/2}}(H_i)\|_{2,\mu_i}^2 \,\mathrm{d}t$$
$$= \left\|H_0^2 - H_i^2\right\|_{1,\mu_i} \le \|H_0 - H_i\|_{2,\mu_i} \|H_0 + H_i\|_{2,\mu_i} .$$

But (d) implies $\|H_0 - H_i\|_{2,\mu_i} \le (6\delta)^{1/2}$ and (a) implies $\|H_0 + H_i\|_{2,\mu_i} \le \|H_0\|_{2,\mu_i} + \|H_i\|_{2,\mu_i} \le \|b_0\|_{2,\mathrm{Tr}} + \|b_i\|_{2,\mathrm{Tr}} = 2$. Thus, by applying (c) to the functions $f_i = \chi_{[t^{1/2},\infty)}$, $0 \le i \le n$, for each $t > 0$, and summing up the above inequalities over i we obtain

$$(*) \qquad \int_{\mathbf{R}_+^*} \sum_{i=1}^{n} \|\Phi_0(e_{t^{1/2}}(b_0)) - \Phi_i(e_{t^{1/2}}(b_i))\|_{2,\mathrm{Tr}}^2 \,\mathrm{d}t$$
$$\le 2n(6\delta)^{1/2} = 2n(6\delta)^{1/2} \int_{\mathbf{R}_+^*} \|e_{t^{1/2}}(b_0)\|_{2,\mathrm{Tr}}^2 \,\mathrm{d}t .$$

This implies that if we denote by D the set of all $t > 0$ for which

$$g(t) \stackrel{\text{def}}{=} \sum_{i=1}^{n} \|\Phi_0(e_{t^{1/2}}(b_0)) - \Phi_i(e_{t^{1/2}}(b_i))\|_{2,\mathrm{Tr}}^2 \,\mathrm{d}t < \delta^{1/4} \|e_{t^{1/2}}(b_0)\|_{2,\mathrm{Tr}}^2 ,$$

then

$$\int_D \|e_{t^{1/2}}(b_0)\|_{2,\mathrm{Tr}}^2 \,\mathrm{d}t \ge 1 - 5n\delta^{1/4} .$$

Indeed, from $\int_D \|e_{t^{1/2}}(b_0)\|_{2,\mathrm{Tr}}^2 \,\mathrm{d}t < 1 - 5n\delta^{1/4}$, by taking into account that $g(t) \ge \delta^{1/4} \|e_{t^{1/2}}(b_0)\|_{2,\mathrm{Tr}}^2$ for $t \in \mathbf{R}_+^* \setminus D$, we would get:

$$\int_{\mathbf{R}_+^*} g(t)\,\mathrm{d}t \ge \int_{\mathbf{R}_+^* \setminus D} g(t)\,\mathrm{d}t$$
$$\ge \delta^{1/4} \int_{\mathbf{R}_+^* \setminus D} \|e_{t^{1/2}}(b_0)\|_{2,\mathrm{Tr}}^2 \,\mathrm{d}t$$
$$\ge 5n\delta^{1/2} > 2n(6\delta)^{1/2} .$$

which is in contradiction with $(*)$.

In particular, since $\delta < (5n)^{-4}$, we have $1 - 5n\delta^{1/4} > 0$ so that $D \neq \varnothing$. Thus, any $s > 0$ with $s^2 \in D$ will satisfy (i).

Proof of (ii) *in the Theorem.* To prove (ii), note first that $\mathrm{Tr} \circ \Phi_0 \le \mathrm{Tr}$ already implies that for each fixed $x \in P_{2+}$ the map $L^1(P_1, \mathrm{Tr}) \ni x_1 \mapsto \mathrm{Tr}(x\Phi_0(x_1))$ defines a positive functional on $L^1(P_1, \mathrm{Tr})$, which we denote by $\Phi_0^*(x)$. Also, if we identify $L^1(P_1, \mathrm{Tr})^*$ with P_1, then $0 \le x \le 1$ implies $0 \le \Phi_0^*(x) \le 1$. Moreover, if in addition we have $\mathrm{Tr} \circ \Phi_0 = \mathrm{Tr}$, then $\Phi_0^*(1) = 1$, so Φ_0^* defines a positive, unital, linear mapping from P_2 into $P_1 = L^1(P_1, \mathrm{Tr})^*$ satisfying $\mathrm{Tr} \circ \Phi_0^* = \mathrm{Tr}$. Consequently, if we denote $\Phi_i' = \Phi_0^* \circ \Phi_i \colon P_1 \to P_1$, $1 \le i \le n$, then $\Phi_i'(1) = 1$, $\mathrm{Tr} \circ \Phi_i' \le \mathrm{Tr}$, $\forall i$, $1 \le i \le n$, and we have the estimates:

$$\begin{aligned}
\|\Phi_i'(b_i) - b_0\|_{2,\mathrm{Tr}}^2 &= \|\Phi_i'(b_i)\|_{2,\mathrm{Tr}}^2 + \|b_0\|_{2,\mathrm{Tr}}^2 - 2\,\mathrm{Tr}(\Phi_0^*(\Phi_i(b_i))b_0) \\
&\le 2 - 2\,\mathrm{Tr}(\Phi_i(b_i)\Phi_0(b_0)) \\
&= 2 + \|\Phi_0(b_0) - \Phi_i(b_i)\|_{2,\mathrm{Tr}}^2 - \|\Phi_0(b_0)\|_{2,\mathrm{Tr}}^2 - \|\Phi_i(b_i)\|_{2,\mathrm{Tr}}^2 \\
&\le 2 + \delta^2 - 2(1-\delta)^2 < 2\delta\,.
\end{aligned}$$

In particular, if we denote $\delta' = (2\delta)^{1/2}$ then the above implies:

$$\|\Phi_i'(b_i)\|_{2,\mathrm{Tr}} \ge \|b_i\|_{2,\mathrm{Tr}} - (2\delta)^{1/2} = 1 - \delta'\,.$$

Altogether, this shows that we can apply the first part of the proof, with $\Phi_0' = \mathrm{id}, \Phi_1', \ldots, \Phi_n'$ instead of $\Phi_0, \Phi_1, \ldots, \Phi_n$ and δ' instead of δ, with the same $b_0, b_1, \ldots, b_n$, to obtain that the set D' of all $t > 0$ for which

$$\sum_{i=1}^{n} \|e_{t^{1/2}}(b_0) - \Phi_i'(e_{t^{1/2}}(b_i))\|_{2,\mathrm{Tr}}^2 < {\delta'}^{1/4} \|e_{t^{1/2}}(b_0)\|_{2,\mathrm{Tr}}^2$$

satisfies

$$\int_{D'} \|e_{t^{1/2}}(b_0)\|_{2,\mathrm{Tr}}^2 \,\mathrm{d}t \ge 1 - 5n{\delta'}^{1/4}\,.$$

Note that for $t \in D'$ we have:

$$\|e_{t^{1/2}}(b_0) - \Phi_i'(e_{t^{1/2}}(b_i))\|_{2,\mathrm{Tr}} < {\delta'}^{1/8} \|e_{t^{1/2}}(b_0)\|_{2,\mathrm{Tr}}$$

for all $i = 1, \ldots, n$. Due to this we also get:

$$\begin{aligned}
\mathrm{Tr}(e_{t^{1/2}}(b_i)) &= \|e_{t^{1/2}}(b_i)\|_{2,\mathrm{Tr}}^2 \\
&\ge \|\Phi_i'(e_{t^{1/2}}(b_i))\|_{2,\mathrm{Tr}}^2 \ge (1 - {\delta'}^{1/8})^2 \|e_{t^{1/2}}(b_0)\|_{2,\mathrm{Tr}}^2 \\
&= (1 - {\delta'}^{1/8})^2\, \mathrm{Tr}(e_{t^{1/2}}(b_0)) > (1 - 2{\delta'}^{1/8})\, \mathrm{Tr}(e_{t^{1/2}}(b_0))
\end{aligned}$$

for all $t \in D'$ and all $i = 1, \ldots, n$. Let then D_i' be the set of all $t \in D'$ for which $\mathrm{Tr}(e_{t^{1/2}}(b_i)) \le (1 + {\delta'}^{1/16})\,\mathrm{Tr}(e_{t^{1/2}}(b_0))$. It follows that we have:

$$\begin{aligned}\delta'^{1/16}\int_{D'\setminus D'_i}\operatorname{Tr}(e_{t^{1/2}}(b_0))\,\mathrm{d}t &\le \int_{D'\setminus D'_i}(\operatorname{Tr}(e_{t^{1/2}}(b_i))-\operatorname{Tr}(e_{t^{1/2}}(b_0)))\,\mathrm{d}t\\ &= \int_{D'}\operatorname{Tr}(e_{t^{1/2}}(b_i))-\operatorname{Tr}(e_{t^{1/2}}(b_0)))\,\mathrm{d}t\\ &\qquad+\int_{D'_i}(\operatorname{Tr}(e_{t^{1/2}}(b_0))-\operatorname{Tr}(e_{t^{1/2}}(b_i)))\,\mathrm{d}t\\ &\le \int_{\mathbf{R}_+^*\setminus D'}(\operatorname{Tr}(e_{t^{1/2}}(b_0))-\operatorname{Tr}(e_{t^{1/2}}(b_i)))\,\mathrm{d}t\\ &\qquad+2\delta'^{1/8}\int_{D'_i}\operatorname{Tr}(e_{t^{1/2}}(b_0))\,\mathrm{d}t\\ &\le \int_{\mathbf{R}_+^*\setminus D'}\operatorname{Tr}(e_{t^{1/2}}(b_0))\,\mathrm{d}t+2\delta'^{1/8}\\ &\le 5n\delta'^{1/4}+2\delta'^{1/8}<3\delta'^{1/8},\end{aligned}$$

in which we used, in the previous estimates, the identity

$$\int_{D'}(\operatorname{Tr}(e_{t^{1/2}}(b_i))-\operatorname{Tr}(e_{t^{1/2}}(b_0)))\,\mathrm{d}t=\int_{\mathbf{R}_+^*\setminus D'}(\operatorname{Tr}(e_{t^{1/2}}(b_0))-\operatorname{Tr}(e_{t^{1/2}}(b_i)))\,\mathrm{d}t$$

(which follows from the equalities

$$\|b_i\|_{2,\mathrm{Tr}}^2=\int_{\mathbf{R}_+^*}\operatorname{Tr}(e_{t^{1/2}}(b_i))\,\mathrm{d}t=\int_{\mathbf{R}_+^*}\operatorname{Tr}(e_{t^{1/2}}(b_0))\,\mathrm{d}t=\|b_0\|_{2,\mathrm{Tr}}^2)$$

and the fact that $5n\delta'^{1/8}<1$.

It thus follows that if we put $D''=\bigcap_{i=1}^n D'_i\cap D$ and take into account that $\delta<(5n)^{-32}$, then we get:

$$\begin{aligned}\int_{D''}\operatorname{Tr}(e_{t^{1/2}}(b_0))\,\mathrm{d}t &\ge \int_{D\cap D'}\operatorname{Tr}(e_{t^{1/2}}(b_0))\,\mathrm{d}t-\sum_{i=1}^n\int_{D'\setminus D'_i}\operatorname{Tr}(e_{t^{1/2}}(b_0))\,\mathrm{d}t\\ &\ge 1-5n\delta^{1/4}-5n\delta'^{1/4}-3n\delta'^{1/16}\\ &>1-5n\delta^{1/32}>0.\end{aligned}$$

Thus $D''\neq\varnothing$. But if $s>0$ is such that $s^2\in D''$ then from the above we have

$$|\operatorname{Tr}(e_s(b_0))-\operatorname{Tr}(e_s(b_i))|<\delta^{1/16}\operatorname{Tr}(e_s(b_0)),$$

which ends the proof of (ii).

Proof of (iii) *in the Theorem.* Finally, (iii) follows now immediately from the last inequality above, since we have:

$$\begin{aligned}\|\Phi_i(e_s(b_i))\|_{2,\mathrm{Tr}} &\geq \|\Phi_0^*(\Phi_i(e_s(b_i)))\|_{2,\mathrm{Tr}}\\ &\geq \|e_s(b_0)\|_{2,\mathrm{Tr}} - \|e_s(b_0) - \Phi_i'(e_s(b_i))\|_{2,\mathrm{Tr}}\\ &> (1-\delta'^{1/8})\|e_{t^{1/2}}(b_0)\|_{2,\mathrm{Tr}}\\ &> (1-\delta'^{1/8})(1+\delta^{1/16})^{-1/2}\|e_{t^{1/2}}(b_i)\|_{2,\mathrm{Tr}}\\ &\geq (1-\delta^{1/32})\|e_{t^{1/2}}(b_i)\|_{2,\mathrm{Tr}}\,.\end{aligned}$$

This ends the proof of the last part of the theorem. □

Proof of Corollary 0.2. As for the Corollary in the Introduction, it follows readily from the Theorem, by taking $n = 1$, $\Phi_0 = \Phi_1 = \Phi$, once we observe that, since Φ is positive, it is selfadjoint, so

$$\begin{aligned}&\sup\left\{ \|\Phi(x)\|_{2,\mathrm{Tr}} \;\middle|\; x \in P_1,\ \|x\|_{2,\mathrm{Tr}} = 1\right\}\\ &\qquad = \sup\left\{ \|\Phi(x)\|_{2,\mathrm{Tr}} \;\middle|\; x \in P_1,\ x = x^*,\ \|x\|_{2,\mathrm{Tr}} = 1\right\},\end{aligned}$$

and also by noticing that if $x \in P_1$ is such that $x = x^*$ then $\|\Phi(|x|)\|_{2,\mathrm{Tr}} \geq \|\Phi(x)\|_{2,\mathrm{Tr}}$. Indeed, this is because by approximating x by step functions (through spectral calculus) we may assume $x = \sum_i c_i p_i$ for some real scalars c_i and finitely many, mutually orthogonal projections of finite trace p_i. Then, taking into account that $\Phi(p_i), \Phi(p_j) \geq 0$ implies $\mathrm{Tr}(\Phi(p_i)\Phi(p_j)) \geq 0$, we get:

$$\begin{aligned}\|\Phi(x)\|_{2,\mathrm{Tr}}^2 &= \textstyle\sum_{i,j} c_i\bar{c}_j \,\mathrm{Tr}(\Phi(p_i)\Phi(p_j))\\ &\leq \textstyle\sum_{i,j} |c_i||\bar{c}_j| \,\mathrm{Tr}(\Phi(p_i)\Phi(p_j)) = \|\Phi(|x|)\|_{2,\mathrm{Tr}}^2\,.\end{aligned}$$

□

2. APPLICATIONS

We shall apply Theorem 0.1 to a case when the semifinite algebras are in fact commutative. We mention that the noncommutativity will be implicitly present though, through the consideration of the positive maps. Note also that in the proof of the Corollary below, only part (i) in the conclusion of the Theorem is being used. In turn, the proof of this part of the Theorem is relatively short.