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FULL REDUCIBILITY AND INVARIANTS FOR $SL_2(\mathbf{C})$

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1. Let G be a group, V a finite dimensional vector space over a commutative field k (mostly $\mathbf{C}$ in this lecture), n the dimension of V and π a representation of G in V i.e. a homomorphism $G \to GL(V)$ of G into the group $GL(V)$ of invertible linear transformations of V. The choice of a basis of V provides an isomorphism of V with k^n, of $GL(V)$ with the group $\mathrm{GL}_n(k)$ of $n \times n$ invertible matrices with coefficients in k, and a realization of π as a matrix representation:

$$g \mapsto \pi(g) = \big(\pi(g)_{ij}\big)_{1 \leq i,j \leq n}. \tag{1}$$

Two main problems pertaining to this situation were considered already in the 19th century, in various special cases, for $k = \mathbf{C}$.

I) INVARIANTS. Let $k[V]$ be the space of polynomials on V with coefficients in k and $k[V]_m$ $(m \in \mathbf{N})$ the space of homogeneous polynomials of degree m. The group G acts via π on $k[V]$ by the rule

$$g \circ P(v) = P\big(\pi(g)^{-1} . v\big) \qquad (v \in V,\ P \in k[V],\ g \in G)$$

leaving each $k[V]_m$ stable. [The argument on the right-hand side will usually be written $g^{-1} . v$ if there is no ambiguity about π.]

Let $k[V]^G$ be the space of polynomials which are invariant under G, i.e. which are constant on the orbits of G. It is an algebra over k and the (first) problem of invariant theory is to know whether it is finitely generated, as a k-algebra.

II) FULL REDUCIBILITY. The representation (π, V) is said to be reducible if there exists a G-invariant subspace $W \neq \{0\}, V$, and fully or completely reducible if any G-invariant subspace has a G-invariant complement. If so, V can be written as a direct sum of G-invariant irreducible subspaces. One is interested in groups having classes of fully reducible representations or in finding families of groups all of whose representations over a given k are fully reducible.

In this lecture, I shall discuss the history of these two problems mainly for one group, namely the group $\mathrm{SL}_2(\mathbf{C})$ of 2×2 complex invertible matrices of determinant one, for $k = \mathbf{C}$ and holomorphic representations, i.e. in which the $\pi(g)_{ij}$ in (1) are holomorphic functions in the entries of g. Occasionally, some remarks will be made on other groups, to put certain results in a more general context, or for historical reasons, but our main focus of attention will still be $\mathrm{SL}_2(\mathbf{C})$. Even so restricted, this history is surprisingly complicated, in part because the principal contributors were sometimes not aware of other work already done. In one case, it seems even that one of them had forgotten some of his own.

2. The irreducible representations of $\mathrm{SL}_2(\mathbf{C})$ were determined by S. Lie. As we know, there is for each $m \in \mathbf{N}$, up to equivalence, one representation of degree $m+1$ in the space, to be denoted V_m, of homogeneous polynomials of degree m on $\mathbf{C}^2$, acted upon via the identity representation of $\mathrm{SL}_2(\mathbf{C})$.

In fact, S. Lie formulated his result differently, more geometrically [LE]. For him, a representation is not a linear one, but a projective one, i.e. a homomorphism into the group of projective transformations of some complex projective space $\mathbf{P}_m(\mathbf{C})$. As usual, $\mathbf{P}_m(\mathbf{C})$ is viewed as the quotient of $\mathbf{C}^{m+1} - \{0\}$ by dilations. This identifies the group $\mathrm{Aut}\,\mathbf{P}_m(\mathbf{C})$ of projective transformations of $\mathbf{P}_m(\mathbf{C})$ with the quotient $\mathrm{GL}_{m+1}(\mathbf{C})/\mathbf{C}^*$ of $\mathrm{GL}_{m+1}(\mathbf{C})$ by the non-zero multiples of the identity matrix, or also with $\mathrm{PSL}_{m+1}(\mathbf{C}) = \mathrm{SL}_{m+1}(\mathbf{C})/\text{center}$, i.e. modulo the group of multiples $c\,.\,\mathrm{Id}$ of the identity matrix, where $c^{m+1} = 1$. Let B be the group of upper triangular matrices in $\mathrm{SL}_2(\mathbf{C})$. The quotient $\mathrm{SL}_2(\mathbf{C})/B = C$ is a smooth complete rational curve, i.e. a copy of $\mathbf{P}_1(\mathbf{C})$. The group B is solvable, connected, therefore, by Lie's theorem it has a fixed point in any projective representation and so, if this point is not fixed under G, its orbit is a copy of C. Lie looks for the cases where such a C is "as curved as possible" ("möglichst gekrümmt") meaning, not contained in a proper projective subspace. It is also a fact, implicitly assumed by Lie, that the action of $\mathrm{SL}_2(\mathbf{C})$ on such a curve is always induced by projective transformations of the ambient projective space. Therefore the search of smooth rational complete curves in projective spaces which are "as curved as possible", up to projective transformations, is tantamount to the classification of the irreducible holomorphic representations of $SL_2(\mathbf{C})$ (linear or projective, there is no essential difference since $SL_2(\mathbf{C})$ is simply connected), up to equivalence. Given $m \geq 1$, the smooth projective rational curves not contained in a proper projective subspace, of smallest degree (number of intersection points with a generic hyperplane), form in $\mathbf{P}_m(\mathbf{C})$ one family, operated upon transitively by $\mathrm{Aut}\,\mathbf{P}_m(\mathbf{C})$, and the degree is m. The irreducible representations are those in which the G-orbit of a fixed point of B has degree m. It is then the only closed orbit of G. In [LE], p. 785–6, S. Lie reports that E. Study has proved the full reducibility of the representations of $SL_2(\mathbf{C})$ (again, in an equivalent projective formulation I do not recall here, but which will appear in § 13), but he does not describe the proof because it is long, maybe not quite correct, and simplifications are hoped for. He adds it is very likely to be true for representations of $SL_n(\mathbf{C})$, any $n \geq 2$. In fact, Study had made this conjecture in a letter to him, even more generally for semisimple groups.

3. In his Thesis E. Cartan provides a proof of full reducibility [Cr1]. It is algebraic, deals with Lie algebras so establishes in fact the full reducibility of the representations of the Lie algebra $\mathfrak{sl}_2(\mathbf{C})$ of $SL_2(\mathbf{C})$, but this is equivalent. He does not state the theorem explicitly, however. The proof is embedded (pp. 100–2) in that of another one, due to F. Engel, to the effect that a non-solvable Lie algebra always contains a copy of $\mathfrak{sl}_2(\mathbf{C})$. But a statement is given at the beginning of Chapter VII (p. 116) with a reference to the passage just quoted for the proof.

In 1896, G. Fano, who knew about Study's theorem through [LE] and was surely not aware of Cartan's proof, maybe not even of Cartan's Thesis, gave an entirely different one in the framework of algebraic geometry, using the properties of "rational normal scrolls" [F].

He first makes two remarks of an algebraic nature which simplify the argument.

a) An induction on the length of a composition series shows it suffices to carry the proof when the space E of the given representation contains one irreducible G-invariant subspace F such that E/F is also irreducible. In other words, since the V_m's are the irreducible representations, up to equivalence, it suffices to consider the case of an exact sequence

$$0 \rightarrow V_m \rightarrow E \rightarrow V_n \rightarrow 0 \tag{1}$$

(again, in projective language, see 13.1).

b) In (1), it may be assumed that $m \geq n$. If $m < n$, this is seen by going over to the contragredient representations

$$0 \to V_n^* \to E^* \to V_m^* \to 0, \tag{2}$$

noting that E is fully reducible if and only if E^* is, and that for each m, the representation V_m is self-contragredient. This also shows that it suffices to consider the case where $m \leq n$. In fact, this last reduction allows for a considerable simplification in Cartan's proof, whereas the reduction to $m \geq n$ is the one Fano uses. (see § 12, § 13 for more details).

4. Another development came from a different source: the idea of averaging over a finite group. In 1896 it was shown that a finite group G of linear transformations always leaves invariant a positive non-degenerate hermitian form. It was stated by A. Loewy without proof [L] and by E.H. Moore, who announced it at some meeting, communicated his proof to F. Klein, and published it later [Mo]. This argument is the now standard one (and Loewy stated later it was his, too): starting from a positive non-degenerate hermitian form $H(\ ,\ )$ on $\mathbf{C}^n$, he considers the sum $H^o(\ ,\ )$ of its transforms under the elements of G:

$$H^o(x,y) = \sum_{g \in G} H(g^{-1}.x\,,\, g^{-1}.y) \tag{1}$$

which he calls a *universal invariant* for G. It is obviously G-invariant and positive non-degenerate. This construction seems quite obvious, but Klein viewed it as interesting enough to make it the subject matter of a communication to the German Math. Soc. [K]. For Moore it was an application of a "well-known group theoretic process". In [Lo] and [Mo], this fact is used to show that a linear transformation of finite order is diagonalizable (which was known, but with more complicated proofs). A bit later, H. Maschke, a colleague of Moore at Chicago, made use of this universal invariant to establish the full reducibility of linear representations of a finite group [Ma]. The standard argument is of course to point out that if V is a G-invariant subspace, then so is its orthogonal complement with respect to H^o. This is the gist of Maschke's proof, but presented in a rather complicated manner.

5. The idea of averaging was pushed much further by A. Hurwitz in a landmark paper [H]. He was interested in the invariant problem. He starts by saying it is well-known one can construct invariants for a finite linear group by averaging, but he is concerned with certain infinite groups, specifically $\mathrm{SL}_n(\mathbf{C})$ and the special complex orthogonal group $\mathrm{SO}_n(\mathbf{C})$ $(n \geq 2)$.

Hurwitz recalls first that if G is a finite linear group acting on $\mathbf{C}^n$ and P is a polynomial on $\mathbf{C}^n$ then the polynomial $P^\natural$ defined by

$$(1) \qquad P^\natural(x) = N^{-1} . \sum_{g \in G} P(g^{-1} . x) \qquad (x \in \mathbf{C}^n),$$

where N is the order of G, is obviously invariant under G (the factor N^{-1} is inserted so that $P^\natural = P$ if P is invariant). If now G is infinite, the initial idea is to replace the summation in (1) by an integration, with respect to a measure invariant by translations. However, if the group is not compact (Hurwitz says bounded), this integral may well diverge. To surmount that difficulty, A. Hurwitz used a procedure which turned out later to be far reaching, namely to integrate over a compact subgroup G_u, which insures convergence, but choosing it big enough so that invariance under G_u implies the invariance under the whole group, an argument later called the "unitarian trick" by H. Weyl [W1]. This is carried out for $SU_n \subset SL_n(\mathbf{C})$ and $SO_n \subset SO_n(\mathbf{C})$. I describe it for $G = SL_2(\mathbf{C})$ and $G_u = SU_2$. The latter is

$$(2) \qquad G_u = SU_2 = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}; \quad a, b \in \mathbf{C}, \quad |a|^2 + |b|^2 = 1 \right\}.$$

Write $a = x_1 + ix_2$, $b = x_3 + ix_4$, with the x_i real. Then SU_2 may be identified to the unit 3-sphere

$$(3) \qquad \mathbf{S}^3 = \{(x_1, \ldots, x_4) \in \mathbf{R}^4, \quad x_1^2 + \cdots + x_4^2 = 1\}.$$

It can be parametrized by the Euler angles φ, ψ, θ:

$$(4) \qquad \begin{aligned} x_1 &= \cos\psi . \cos\varphi . \cos\theta \\ x_2 &= \cos\psi . \cos\varphi . \sin\theta \\ x_3 &= \cos\psi . \sin\varphi \\ x_4 &= \sin\psi . \end{aligned} \qquad (|\varphi|, |\psi| \leq \pi/2, \ \theta \in [0, 2\pi])$$

The measure

$$(5) \qquad dv = \cos\psi . \cos\varphi . d\psi . d\varphi . d\theta$$

is then invariant under translations and the volume of $\mathbf{S}^3$ with respect to dv is 8π. Let $\sigma: G \to GL_N(\mathbf{C})$ be a holomorphic linear representation of G and P be a polynomial on $\mathbf{C}^N$. Integration on G_u yields the polynomial $P^\natural$ given by

$$(6) \qquad \begin{aligned} P^\natural(x) &= (8\pi)^{-1} \int_{G_u} P(g^{-1} . x)\, dv \\ &= (8\pi)^{-1} \int_{-\pi/2}^{\pi/2} \cos\psi . d\psi \int_{-\pi/2}^{\pi/2} \sin\varphi . d\varphi \int_0^{2\pi} P(g^{-1} . x)\, d\theta \end{aligned}$$

($x \in \mathbf{C}^N$). It is invariant under the action of G_u and the claim is that it is even invariant under G itself. Given $x \in \mathbf{C}^N$ consider the function μ_x on G given by

$$\mu_x(g) = P^\natural(g^{-1}.x) - P^\natural(x) \qquad (g \in G). \tag{7}$$

It is holomorphic in the entries of g, and is identically zero for $g \in G_u$. To establish that it is identically zero on G, it suffices to show that it is zero on a neighborhood U of the identity. The tangent space to G (resp. G_u) at the identity is the complex (resp. real) vector space $\mathfrak{g}$ (resp. $\mathfrak{g}_u$) of 2×2 complex (resp. skew-hermitian) matrices of trace zero. Take U small enough so that it is the isomorphic image of a neighborhood U_o of the origin in $\mathfrak{g}$ under the exponential mapping. Let $\tilde{\mu}_x$ be the pull back of $\mu_x|_U$ by the inverse mapping. Then $\tilde{\mu}_x$ is a holomorphic function on U_o which is zero on $U_o \cap \mathfrak{g}_u$. But $\mathfrak{g}_u$ is a real form of $\mathfrak{g}$, i.e. as a real vector space, $\mathfrak{g}$ is the direct sum of $\mathfrak{g}_u$ and of the space $i\mathfrak{g}_u$ of hermitian 2×2 matrices of trace zero. Hence $\tilde{\mu}_x$ is identically zero on U_o and our assertion follows.

6. From this Hurwitz deduces that the algebra $\mathbf{C}[\mathbf{C}^N]^G$, to be denoted I_G to simplify notation, of invariant polynomials on $\mathbf{C}^N$ is finitely generated: The projector $P \mapsto P^\natural$ obviously satisfies the relation

$$(P.Q)^\natural = P.Q^\natural, \quad \text{if } P \text{ is } G\text{-invariant}. \tag{1}$$

By Hilbert's finiteness theorem, the ideal I generated by the elements of I_G without constant term is finitely generated. Let $Q_i \in I_G$ $(1 \leq i \leq s)$ be a generating system of this ideal, which may be assumed to consist of homogeneous invariant elements of strictly positive degrees. Let now $Q \in I_G$ be homogeneous. It certainly belongs to I. There exist therefore homogeneous polynomials $A_1, \ldots, A_s$ such that

$$Q = \sum_{1 \leq i \leq s} Q_i . A_i .$$

Then, we have, by (1)

$$Q^\natural = Q = \sum_i Q_i . A_i^\natural .$$

Since the $A_i^\natural$ have strictly lower degrees than Q, it follows by induction on the degree, that I_G is generated, as an algebra, by the Q_i.

In analogy with Maschke's theorem, Hurwitz could have easily given a new proof of the full reducibility of the holomorphic representations of $\mathrm{SL}_2(\mathbf{C})$, and, more generally the first proof for $\mathrm{SL}_n(\mathbf{C})$ $(n \geq 3)$ and $\mathrm{SO}_n(\mathbf{C})$ $(n \geq 4)$.

Indeed if, as in §4, $H(\ ,\ )$ is a positive non-degenerate hermitian form on $\mathbf{C}^N$, the form H^o constructed as in (1), but using integration

$$H^o(x,y) = \int_{G_u} H(g^{-1}.x, g^{-1}.y)\,dv \qquad (x,y \in \mathbf{C}^N)$$

is invariant under G_u and still positive non-degenerate, whence the full reducibility of the (continuous) representations of G_u. There remains to show that every G_u-invariant subspace is G-invariant. Let V be one and W its orthogonal complement with respect to H^o. Fix a basis $(f_1,\dots,f_N)$ of $\mathbf{C}^N$ whose first $p = \dim V$ elements span V and the last $N-p$ span W. Then the matrix coefficients $\sigma(g)_{ij}$ $(i \le p,\ j > p)$ are holomorphic functions on G which vanish on G_u hence, by the argument outlined previously, are identically zero on G. Therefore V and W are G-invariant, and full reducibility is proved.

7. I spoke of a "landmark paper". This is only by hindsight because the paper was completely forgotten for about 25 years and, apparently, no specialist of Lie groups or Lie algebras was aware of it and had realized that a proof of Study's conjecture for $SL_n(\mathbf{C})$ was at hand.

Meanwhile, progress was made on two fronts:

a) Character theory for complex representations of finite groups, orthogonality relations, etc (Frobenius, Schur, Burnside, 1896-1906).

b) Construction of all irreducible representations of complex simple Lie algebras by E. Cartan ([Cr2], 1914).

In 1922, I. Schur discovers Hurwitz's paper and uses it to extend the character theory a) to representations of SU_n or SO_n [S]. Two years later, H. Weyl combines b) and the point of view of Hurwitz-Schur to generalize it to all complex or compact semisimple groups [W]. Until he came on the scene, Schur was not aware of Cartan's work nor Cartan of Schur's or Hurwitz's. He also points out a gap in [Cr2]: the construction of irreducible representations makes implicit use of full reducibility, a problem Cartan had not considered at all there. At that point, as a proof, there was then only Weyl's generalization of Hurwitz and Schur, which was highly transcendental. Both Cartan and Weyl felt that an algebraic proof of such a purely algebraic statement was desirable, but viewed it as rather unlikely that one would be forthcoming. Cartan could have pointed out that in the case of $SL_2(\mathbf{C})$ or rather its Lie algebra, one was contained in his Thesis. The fact that he did not makes me think that he had forgotten about it (but not forever, though: it is again given in his book on spinors [Cr3]).

8. In comparing physicists and mathematicians it is often said that the physicists, unlike mathematicians, do not care that much for rigorous proofs. Here, we are dealing with the search for a second proof, in a different framework, of a theorem already established, a problem which would normally seem even less attractive to a physicist. In that case, however, it did attract one, H. L Casimir, whose approach had its origin in the use of group representations in quantum mechanics. Since it involves SO_3 or $SO_3(\mathbf{C})$ rather than SU_2 or $SL_2(\mathbf{C})$, let me recall first that SO_3 (resp. $SO_3(\mathbf{C})$) is the quotient of SU_2 (resp. $SL_2(\mathbf{C})$) by its center, which consists of $\pm\,\mathrm{Id}$. In particular, $\mathfrak{sl}_2(\mathbf{C})$ may be viewed as the complexification of the Lie algebra $\mathfrak{so}_3$ of SO_3, so that we can take as a basis of it the infinitesimal rotations D_x, D_y, D_z around the three coordinate axes in $\mathbf{R}^3$, where x, y, z are the coordinates:

$$(1)\quad D_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad D_y = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad D_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Viewed as differential operators on functions, these transformations are

$$(2)\qquad D_x = y\,.\,\partial_z - z\,.\,\partial_y, \quad D_y = z\,.\,\partial_x - x\,.\,\partial_z, \quad D_z = y\,.\,\partial_x - x\,.\,\partial_y\,.$$

The application to quantum mechanics makes use of

$$L_x = i^{-1}\,.\,D_x, \quad L_y = i^{-1}\,.\,D_y, \quad L_z = i^{-1}\,.\,D_z$$

called the components of the moment of momentum and of

$$(3)\qquad L^2 = L_x^2 + L_y^2 + L_z^2\,,$$

the square of the moment of momentum.

The decisive idea is to use L^2. It is a differential operator, also represented by minus the sum of the square of the matrices in (1). It belongs to the associative algebra of endomorphisms of $\mathbf{C}^2$ generated by $\mathfrak{sl}_2(\mathbf{C})$, a quotient of the so-called universal enveloping algebra of $\mathfrak{sl}_2(\mathbf{C})$, but not to the Lie algebra itself.

An elementary computation shows that L^2 commutes with the infinitesimal rotations, hence with $\mathfrak{sl}_2(\mathbf{C})$ itself. A linear representation (σ, V) extends to one of the enveloping algebra and in particular $\sigma(L^2)$ is defined. If σ is irreducible, then $\sigma(L^2)$ is a scalar multiple of the identity (Schur's lemma).

In the representation V_n of degree $n+1$, this scalar is equal to $n(n+2)/4$. It characterizes the representation, up to equivalence [1]).

In order to prove full reducibility, Casimir notes that it suffices to consider the case of the exact sequence (1) in §3. Assume first $m \neq n$, the main case. Then $\sigma(L^2)$ has two eigenvalues, $m(m+2)/4$ and $n(n+2)/4$. The eigenspace W for the latter eigenvalue intersects V_m only at the origin. Since $\sigma(L^2)$ commutes with $\sigma\big(\mathfrak{sl}_2(\mathbf{C})\big)$, the space W is also invariant under $SL_2(\mathbf{C})$. Its projection in V_n is invariant, non-zero, hence equal to V_n, so W is the sought for complement to V_m. If $m = n$, the existence of an invariant complement is proved by a rather elementary computation, sketched in 12.4.

9. An analog of L^2 had been introduced in 1931 by Casimir for any complex semisimple Lie algebra, later called the Casimir operator. Using it van der Waerden generalized Casimir's argument to give the first algebraic general proof of the full reducibility of finite dimensional representations of complex semisimple Lie algebras [CW].

Later it was realized that the Casimir operator is an element in the center of the universal enveloping algebra (which generates it for $\mathfrak{sl}_2(\mathbf{C})$). The full center was investigated in the late forties by G. Racah, also a physicist, on the one hand, by C. Chevalley and Harish-Chandra on the other, and became a powerful tool in the study of the topology of compact Lie groups and of infinite dimensional representations of semisimple Lie groups.

Racah's motivation was representation theory. From a physicist's point of view, the eigenvalue of L^2 gave a parametrization of an irreducible representation of $SL_2(\mathbf{C})$ by a number with a physical meaning, whereas the highest weight had none. For higher dimensional groups, the eigenvalue of L^2 does not characterize the representation, up to equivalence, which makes the general argument in [CW] quite complicated. Racah's idea was to search for more operators commuting with the Lie algebra (r independent ones if r is the rank of the Lie algebra), the eigenvalues of which would again characterize the irreducible representations. This would then allow one to treat the case of two inequivalent irreducible representations in a short exact sequence in the same way as for $\mathfrak{sl}_2(\mathbf{C})$ and considerably simplify the proof. At that time, the mathematicians were not searching for a new algebraic proof, however, and this was not at all a motivation for Chevalley and Harish-Chandra.

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[1]) In the physics literature and in [W2], the irreducible representations of $SL_2(\mathbf{C})$ are parametrized by $(1/2)\mathbf{N}$. The representation V_j there is our V_{2j}. It has degree $2j+1$ and the eigenvalue of L^2 is $j(j+1)$. It is a spin representation, i.e. non trivial on the center, if and only j is a half-integer.

10. The paper [CW] was followed shortly by two other algebraic proofs, one by R. Brauer [Br] and one based on a lemma of J. H. C. Whitehead, which is now best expressed in the framework of Lie algebra cohomology, and became the standard algebraic argument for a number of years.

In 1956, a new proof was published by P. K. Raševskiĭ [R]. Consider the group $\mathrm{Aff}(\mathbf{C}^N)$ of *affine* transformations of $\mathbf{C}^N$. It is the semidirect product of the group of translations by the group $\mathrm{GL}_N(\mathbf{C})$. Accordingly, its Lie algebra is the semidirect product $\mathfrak{s} \oplus \mathfrak{t}$ of the space of translations $\mathfrak{t}$ by the Lie algebra $\mathfrak{s}$ of $\mathrm{GL}_N(\mathbf{C})$. The new ingredient is the proof that any representation of a semisimple Lie algebra in the Lie algebra $\mathfrak{aff}(\mathbf{C}^N)$ of $\mathrm{Aff}(\mathbf{C}^N)$ leaves a point of $\mathbf{C}^N$ fixed or, globally speaking, any complex semisimple group of affine transformations of $\mathbf{C}^N$ has a fixed point. Let now σ be a representation of the complex semisimple Lie algebra $\mathfrak{g}$ in $\mathbf{C}^M$ and $V \subset \mathbf{C}^M$ an invariant subspace. Then the set of subspaces W of $\mathbf{C}^M$ complementary to V forms in a canonical way an affine space, with space of translations $\mathbf{C}^M/V$. It is operated upon naturally by $\sigma(\mathfrak{g})$. The existence of a fixed point implies that of a $\mathfrak{g}$-invariant complement to V, whence the full reducibility.

When N. Bourbaki was preparing Volume 1 of the book on Lie groups and Lie algebras, entirely devoted to Lie algebras, an algebraic proof was needed. The cohomological one did not seem really suitable, requiring as it did lots of preliminaries on cohomology of Lie algebras, which it did not seem appropriate to introduce at that early stage of the exposition. Then Bourbaki turned to Raševskiĭ's proof and made it somewhat more algebraic and self-contained. After the book was published in 1961, I stumbled once on a copy of [Br], and realized this argument was the one of [Br], another example of a paper overlooked for over 25 years, the knowledge of which would have saved some work to Bourbaki.

11. This pretty much concludes my story, but as Poincaré once wrote, there are no problems which are completely solved, only problems which are more or less solved. Still considering SL_2 one may ask about the problems I and II for $\mathrm{SL}_2(k)$ where k is an algebraically closed groundfield of positive characteristic p. It is well-known that full reducibility does not necessarily hold. Take for example k of characteristic two and the representation V_2 of degree three on the homogeneous quadratic polynomials. It has $x^2, x.y$ and y^2 as a basis. In characteristic 2, we have the rule $(a+b)^2 = a^2 + b^2$ so the linear combinations of x^2 and y^2 are the squares of the linear forms, and form a two-dimensional invariant subspace V. A complementary subspace is of dimension one therefore, if invariant, would be acted up trivially by $\mathrm{SL}_2(k)$.

However, $SL_2(k)$ does not leave any non-zero element of V_2 stable, so V_2 is not fully reducible.

This does not rule out a positive answer to problem I, but if so, another approach had to be devised. D. Mumford proposed a weaker notion than full reducibility, now called geometric reductivity: if C is an invariant one-dimensional subspace, there exists a homogeneous G-invariant hypersurface not containing C (in the case of full reducibility it could be a hyperplane). Then Nagata showed that this condition indeed implies the finite generation of the algebra of invariants. Later geometric reductivity was proved by C.S. Seshadri for $SL_2(k)$ and by W. Haboush in general.

Even over $\mathbf{C}$, the problems of full reducibility and of the determination of irreducible representations resurfaced not for $SL_2(\mathbf{C})$, but for its generalization as a Kac-Moody Lie algebra, or for the deformation of its Lie algebra as a "quantum group". This has led to further problems and to more contacts with mathematical physics.

APPENDIX: MORE ON SOME PROOFS OF FULL REDUCIBILITY

We give here more technical details on the proofs of full reducibility for $\mathfrak{sl}_2(\mathbf{C})$ or $SL_2(\mathbf{C})$ due to Cartan, Fano and Casimir, assuming some familiarity with Lie algebras and algebraic geometry. We let $\mathfrak{g}$ stand for $\mathfrak{sl}_2(\mathbf{C})$.

12. LIE ALGEBRA PROOF:

12.1. Let

(1) $$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}.$$

be the familiar basis of $\mathfrak{g}$. It satisfies the relations

(2) $$[h,e] = 2e \qquad [h,f] = -2f \qquad [e,f] = -h\,.$$

The elements h, e, f define one-parameter subgroups ($t \in \mathbf{R}$)

$$e^{th} = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \quad e^{te} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad e^{tf} = \begin{pmatrix} 1 & 0 \\ -t & 0 \end{pmatrix}.$$

By letting them act on functions of x, y and taking the derivatives for $t = 0$, we get expressions of h, e, f as differential operators, namely

(3) $$h = x\,.\,\partial_x - y\,.\,\partial_y, \quad e = x\,.\,\partial_y, \quad f = -y\,.\,\partial_x\,.$$

Let E be a representation space for $\mathfrak{g}$ and E_c ($c \in \mathbf{C}$) the eigenspace for h with eigenvalue c. Then (2) implies

$$(4) \qquad e \,.\, E_c \subset E_{c+2} \qquad f \,.\, E_c \subset E_{c-2}\,.$$

More generally, if $(h - c\,.\,I)^q \,.\, v = 0$ for some $q \geq 1$, then

$$(5) \qquad (h-(c+2)\,.\,I)^q \,.\, e \,.\, v = 0 = (h-(c-2)\,.\,I)^q \,.\, f \,.\, v = 0\,.$$

12.2. We now consider V_m. It has a basis $x^{m-i} \,.\, y^i$ ($i = 0, \ldots, m$) and $x^{m-i} \,.\, y^i$ is an eigenvector for h, with eigenvalue $m - 2i$. Let

$$(1) \qquad v_{m-2i} = \binom{m}{i} x^{m-i} \,.\, y^i \qquad (i = 0, \ldots, m)\,.$$

The v_{m-2i} form a basis of V_m and we have:

$$(2) \qquad h \,.\, v_{m-2i} = (m-2i) v_{m-2i} \qquad (i = 0, \ldots, m)\,.$$

A simple computation, using 12.1(2), (3), yields

$$\begin{aligned} (3) \qquad & f \,.\, v_{m-2i} = -(i+1)\, v_{m-2i-2} \\ (4) \qquad & e \,.\, v_{m-2i} = (m-i+1)\, v_{m-2i+2} \end{aligned} \qquad (i = 0, \ldots, m)\,,$$

with the understanding that

$$(5) \qquad v_{m+2} = v_{-m-2} = 0\,.$$

(3) and (4) imply

$$\begin{aligned} (6) \qquad & f \,.\, e \,.\, v_{m-2i} = -i(m-i+1) v_{m-2i} \\ (7) \qquad & e \,.\, f \,.\, v_{m-2i} = (i+1)(m-i) v_{m-2i}\,. \end{aligned}$$

REMARKS. (a) The eigenvalues of h in V_m are integers. By consideration of a Jordan-Hölder series, it follows that this is true for any finite dimensional representation.

(b) In $\mathbf{P}(V_m)$ the rational normal curve occurring in Lie's description of the irreducible projective representations of $\mathrm{SL}_2(\mathbf{C})$ (see § 2) is the orbit of the point representing the line spanned by x^m. This is also the unique fixed point in $\mathbf{P}(V_m)$ of the group U generated by e, i.e. the group of upper triangular unipotent (eigenvalues equal to one) matrices. It is therefore also the locus of the fixed points of the conjugates of U in $\mathrm{SL}_2(\mathbf{C})$, and each such conjugate has a unique fixed point in $\mathbf{P}(V_m)$.

12.3. Note that 12.2(5) is a consequence of 12.2(4) and of the commutation relations 12.1(1). A similar argument shows more generally that if E is a representation of $\mathfrak{g}$ and $v \in E$ satisfies the conditions

$$(1) \qquad e.v = 0, \quad h.v = c.v \qquad (c \in \mathbf{C}),$$

then the elements $f^i.v$ ($i \geq 0$) span a finite dimensional $\mathfrak{g}$-submodule F. In particular $\mathbf{C}.v$ is the eigenspace with eigenvalue c and all other eigenvalues of h in F are of the form $c - q$ ($q \in \mathbf{N}$, $q \geq 1$).

12.4. *First proof of full reducibility.* We use 12.3, which is contained in [Cr1] and the two remarks a) and b) of §3. This reduces the proof of full reducibility of a $\mathfrak{g}$-module E to the case of a short exact sequence

$$(1) \qquad 0 \to V_m \to E \xrightarrow{\pi} V_n \to 0 \qquad (m \leq n).$$

Let $m < n$. Then h has an eigenvector $v \in E$ with eigenvalue n, which does not belong to V_m. It is annihilated by e, since there are no weights $> n$ in V_m or V_n, hence in E. By 12.3, it generates a $\mathfrak{g}$-submodule distinct from V_m, which must therefore be a $\mathfrak{g}$-invariant complement to V_m.

Let now $m = n$. Let $\{v_{m-2i}\}$ ($i = 0, \ldots, m$) be the basis of V_m, viewed as subspace of E, constructed in 12.2. Let v'_m be a vector which maps under π onto the similar basis element of the quotient and let $v'_{m-2i} = \binom{m}{i}.v'_m$. Then the v'_{m-2i} project onto the basis of E/V_m defined in 12.2. There exists $a \in \mathbf{C}$ such that

$$(2) \qquad h.v'_m = m.v'_m + a.v_m.$$

We claim it suffices to show that $a = 0$. Indeed, in that case, 12.3 again implies that v'_m generates a $\mathfrak{g}$-submodule distinct from V_m, hence a supplement to V_m.

There remains to prove that $a = 0$. We claim first

$$(3) \qquad h.v'_{m-2i} = (m-2i)v'_{m-2i} + a.v_{m-2i} \qquad (i = 0, \ldots, m).$$

For $i = 0$, this is (2). Assuming it is proved for i, we obtain (3) for $i+1$ by applying f to both sides and using 12.1(2), 12.2(3).

For $i \geq 1$, we have, by 12.1(2) and 12.2(3)

$$(4) \qquad i.e.v'_{m-2i} = -e.f.v'_{m-2i+2} = -f.e.v'_{m-2i+2} + h.v'_{m-2i+2}.$$

By (3) and 12.2(6), this yields

$$(5) \qquad i.e.v'_{m-2i} = i(m-i+1).v'_{m-2i+2} + a.v_{m-2i+2}.$$

If we apply (5) for $i = m+1$, we get $a.v_{-m} = 0$, hence $a = 0$.

REMARK. This last computation is contained in [CW] and also, unknown to the authors, in [Cr1]. As we saw, the proof for $m < n$ reduces immediately to 12.3, and by b) in § 3 it suffices to consider that case when $m \neq n$. A direct computation along the lines of the previous proof is longer if $m > n$ (see 12.5). Cartan performs it even for a Jordan-Hölder series of any length, which leads to a rather complicated argument. By using his operator, Casimir did not have to make any distinction between the cases $m < n$ and $m > n$.

12.5. To give a better idea of Cartan's proof, we discuss the case $m > n$ directly, without reducing to $m < n$.

We let v'_n and v'_{n-2i} $(i \geq 0)$ be as before. Note first that if n and m have different parities, then V_n and V_m have no common eigenvalue for h. In particular h has no element of weight $n+2$ in E and the eigenspace for n is one-dimensional, hence spanned by v'_n. Again, by 12.3, v'_n generates a complementary $\mathfrak{g}$-module. So we assume that $m \equiv n \mod 2$. As before, the whole point is to find v'_n satisfying the condition 12.3(1), for $c = n$.

As above, there is a constant a such that

$$h \,.\, v'_n = n \,.\, v'_n + a \,.\, v_n \,. \tag{1}$$

We want to prove v'_n may be chosen so that $a = 0$. As in 12.4, we see that

$$h \,.\, v'_{n-2i} = (n-2i) \,.\, v'_{n-2i} + a \,.\, v_{n-2i} \qquad (i \geq 0)\,. \tag{2}$$

The weights in V_n are contained in $[n, -n]$, so the projection of $f \,.\, v'_{-n}$ in V_n is zero and we have, for some constant c,

$$f \,.\, v'_{-n} = c \,.\, v_{-n-2} \,. \tag{3}$$

Let $v''_n = v'_n - c \,.\, v_n$ and following 12.2(4), define v''_{n-2i} inductively by the relation

$$v''_{n-2i} = -i \,.\, f \,.\, v''_{n-2i+2} \qquad (i = 1, \ldots, n)\,. \tag{4}$$

By induction on i, we see that

$$h \,.\, v''_{n-2i} = (n-2i) \,.\, v''_{n-2i} + a \,.\, v_{n-2i} \qquad (i = 0, \ldots, n) \tag{5}$$

and also, in view of (3), that

$$f \,.\, v''_{-n} = f \,.\, v'_{-n} - c \,.\, f \,.\, v_{-n} = 0\,. \tag{6}$$

For $i = n$, the equality (5) gives

$$h \,.\, v''_{-n} = -n \,.\, v''_{-n} + a \,.\, v_{-n} \,. \tag{7}$$

Apply now f to both sides and recall that $f.h = h.f + 2f$. In view of (6) and 12.2(3) for $m = n$, we get

(8) $$a.(n+1).v_{-n-2} = 0.$$

But $n < m$ so $v_{-n-2} \neq 0$, whence $a = 0$.

We may therefore assume that v'_n is an eigenvector of h. There is no eigenspace for h with eigenvalue $n+2$ in V'_n, hence

(9) $$e.v'_n = b.v_{n+2} \qquad (b \in \mathbf{C}).$$

By 12.2(4), for $i = (m-n)/2$ (recall that $m \equiv n$ (2)), we get

(10) $$e.v_n = \big((m+n+2)/2\big).v_{n+2},$$

therefore

(11) $$w_n = v'_n - b\big((m+n+2)/2\big)^{-1}.v_n$$

satisfies the conditions

(12) $$h.w_n = n.w_n, \quad e.w_n = 0,$$

so that, by 12.3, $\mathfrak{g}.w_n$ is a copy of V_n complementary to V_m.

13. Fano's proof:

It deals with projective transformations and uses algebraic geometry. Given a finite dimensional vector space F over $\mathbf{C}$, we let $\mathbf{P}(F)$ be the projective space of one-dimensional subspace of F. If F is of dimension n, $\mathbf{P}(F)$ is isomorphic to $\mathbf{P}_{n-1}(\mathbf{C})$.

13.1. The proof is contained in §§7, 8, 9 of [F]. §9 shows how to reduce it to the case considered in §3, a), b), that is, to the case of a short exact sequence 12.4(1) with $m \geq n$, but expressed in projective language, namely:

The space $\mathbf{P} = \mathbf{P}(E)$ contains a minimal irreducible invariant projective subspace $W = \mathbf{P}(V_m)$ of dimension m and the induced projective representation in the space W' of projective $(m+1)$-subspaces containing W is irreducible.

The problem is then to find an invariant projective subspace D not meeting W. If so, it has necessarily dimension n and $\mathbf{P}(E)$ is the join of W and D. Moreover, by the remark b) in §3, it may be assumed that $m \geq n$. Let us write N for the dimension of $\mathbf{P}$. Then $N = m+n+1$ and $m \geq (N-1)/2$.

As in 12.2(b), U is the one-parameter subgroup of $G = SL_2(\mathbf{C})$ generated by e. Its fixed point set is also the subspace E^e of E annihilated by e. Since

U is unipotent, any line invariant under U is pointwise fixed, so that the projective subspace $\mathbf{P}(E^U)$ associated to E^U is also the fixed point set $\mathbf{P}(E)^U$ of U on $\mathbf{P}(E)$. Similarly, it may be identified with the set $\mathbf{P}(E)^e$ of zeros of the vector field on $\mathbf{P}(E)$ defined by the action of U.

In §7, Fano proves that $\mathbf{P}(E)^e$ is a projective line. I am not sure I understand his argument, so I shall revert to the linear setup. As just pointed out, we have to show that E^e is two-dimensional.

In V_m and V_n it is one-dimensional, so the exact sequence 12.4(1) shows that $\dim E^e \leq 2$. As in §3, let E^* be the contragredient representation to E. Then ${E^*}^e$ is the dual space to E/eE so it is equivalent to prove that $\dim {E^*}^e = 2$. Therefore we may assume that $m \leq n$ (our assumption earlier, but not the one of Fano). Fix a vector $v' \in E$ projecting onto a highest weight vector in V_n. It is an eigenvector of h if $m < n$, is annihilated by $(h - n.I)^2$ otherwise, and in both cases is annihilated by e (see 12.1 (4), (5)).

13.2. The next and main part of Fano's argument depends on some properties of the "rational normal scrolls", which we now recall (see [GH], p. 522–527). Assume $N \geq 2$ and let Z be a surface in $\mathbf{P}$, not contained in any projective subspace. Then its degree is at least $N-1$ ([GH], p. 173). Those of degree $N-1$ have been classified, up to projective transformations ([GH], *loc.cit.*). Only one is not ruled, the Veronese embedding of $\mathbf{P}_2(\mathbf{C})$ in $\mathbf{P}_5(\mathbf{C})$.

The others are the *rational normal scrolls* $S_{a,b}$ $(a+b = N-1)$, obtained in the following way: Fix two independent projective subspaces A, B of dimension a, b. Then $\mathbf{P} = A * B$ is the join of A and B. Let C_A (resp. C_B) be a rational normal curve in A (resp. B) and $\varphi : C_A \to C_B$ an isomorphism. Then $S_{a,b}$ is the space of the lines $D(x, \varphi(x))\,(x \in C_A)$. If $a > 0$, but $b = 0$, then C_B is a point, φ maps C_A onto a point and $S_{a,b}$ is the cone over C_A with vertex C_B. It has a unique singular point, namely C_B and this is the only case where $S_{a,b}$ is not smooth ([GH], p. 525).

A rational curve in $S_{a,b}$ which cuts every line $D(x, \varphi(x))$ in exactly one point is called a directrix. By construction C_A (resp. C_B) is a directrix of degree a (resp. b). The main result used by Fano is that if $a > b$, then C_B is the unique directrix of degree b ([GH], p. 525). Fano deduces this essentially from an earlier result of C. Segre [Se].

If $a = b$, then we may identify A to B by a map φ which takes C_A to C_B. It is clear that in that case $S_{a,b} = S_{a,a} = \mathbf{P}^1(\mathbf{C}) \times \mathbf{P}^1(\mathbf{C})$.

13.3. We now come back to the situation in 13.1. In W there is exactly one rational normal curve C stable under G. The zero set $\mathbf{P}(E)^e$ of e is a line (13.1) and $\mathbf{P}(E)^e \cap C$ consists of one point, namely, W^e. Let Z be the set of transforms $g \cdot \mathbf{P}(E)^e$ of $\mathbf{P}(E)^e$, $(g \in G)$. Since $\mathbf{P}(E)^e$ is stable under the upper triangular group B and G/B is complete (in fact a smooth rational curve), Z is a projective subvariety, a G-stable ruled surface. We first dispose of a special case. The line $g\,.\,P(E)^e$ is the fixed point set of the subgroup ${}^gU = g\,.\,U\,.\,g^{-1}$, conjugate to U by g. Assume that two distinct such lines have a common point. It would then be fixed by two distinct conjugates of U. But it is immediate that two such subgroups generate G, so that there would be a fixed point D of G in $\mathbf{P}(E)$, necessarily outside W. Then $\mathbf{P}(E)$ would be the join of W and D, and we would be through. From now on, we assume that the lines $g\,.\,\mathbf{P}(E)^e$ either coincide or are disjoint. We want to prove that Z has degree $N-1$ in $\mathbf{P}(E)$. First we claim that it is not contained in any hyperplane Y of $\mathbf{P}(E)$. Indeed, if it were, it would be contained in a G-stable proper subspace F, the intersection of the transforms of Y. The subspace F would contain W properly, which would contradict the irreducibility of the quotient representation in $\mathbf{P}(V_n)$. The degree of Z is therefore at least $N-1$ ([GH], p. 173–4). There remains to show that it is $\leq (N-1)$.

Let $C' \subset W'$ be the closed orbit of G, which plays the same role as C in W. In particular, it has degree n. Let now Y be a generic hyperplane of $\mathbf{P}(E)$ among those containing W. Viewed as a hyperplane in W', it cuts C' in n distinct points Q_i $(i = 1, \dots, n)$. Let U_i be the conjugate of U which fixes Q_i (see 12.2, (b)). The intersection $Z \cap Y$ is a (reducible) curve. We want to prove it has degree $N-1$ in Y. We claim first

$$(1) \qquad Y \cap Z = C \cup D_1 \cup \cdots \cup D_n \qquad (D_i = \mathbf{P}(E)^{U_i}),$$

where the D_i are disjoint projective lines, each intersecting C at exactly one point.

First, by construction, $C \subset Z \cap Y$, in fact $C = W \cap Z \cap Y$. Let $x \in Z \cap Y$, $x \notin W$. It belongs to some line $D_g = g\,.\,\mathbf{P}(E)^e$. The line D_g also contains $g\,.\,W^e$, which belongs to $Z \cap Y$, too. Therefore $D_g \subset Y$, and of course $D_g \subset Z$, hence $Z \cap Y$ is the union of C and some of the lines D_g. The line D_g spans with W a projective subspace of dimension equal to $\dim W + 1$, which represents a point of W', fixed under ${}^gU$. It belongs therefore to Y if and only ${}^gU$ is one of the U_i, i.e. if and only if D_g is one of the D_i's and (1) follows.

Since C has degree $m = \dim W$ in W, it follows that $Z \cap Y$ is a curve of degree $m + n$ in Y, hence Z is a surface of degree at most $m + n = N - 1$ in $\mathbf{P}(E)$.

Thus Z is a ruled surface, not contained in a hyperplane, of smallest possible degree. It is therefore a "rational normal scroll" (13.2). It is isomorphic to $S_{a,b}$ where $a = \dim W = m$ and $b = N - 1 - a = n$.

Recall that we have reduced ourselves to the case $a \geq b > 0$. Assume first $a > b$. Then, (see 13.2), Z contains a *unique* directrix of degree b. It is a normal curve in a b-dimensional subspace, which must be invariant under G, since Z is. This provides the complementary subspace to W.

Let now $m = n$. Then (13.2), $Z = C \times C'$ is a product of two copies of $\mathbf{P}^1(\mathbf{C})$, where C is, as before, a G-stable rational normal curve in W and $C' = \mathbf{P}(E)^e$. The transforms $g\,.\,\mathbf{P}(E)^e$ of C' are the lines $\{c\} \times C'\,(c \in C)$.

The lines $C_y = C \times \{y\}$ $(y \in C')$ are "directrices". We claim that they are all invariant under G. Clearly, the intersection number $C_y \cdot C_z$ is zero if $y \neq z$ $(y, z \in C')$. Let $g \in G$. Since it is connected to the identity, we have then also $(g\,.\,C_y) \cdot C_z = 0$, therefore $g\,.\,C_y \cap C_z = \varnothing$ unless $g\,.\,C_y = C_z$. Since $g\,.\,C_y$ must meet at least one C_z, we have $g\,.\,C_y = C_z$ for some z and we see that G permutes the curves $\{C_y\}\,(y \in C')$. Each such curve contains a fixed point of e, hence of U. Therefore C_y is stable under U. Now the subgroup H of G leaving each curve C_y stable is a normal subgroup, which is $\neq \{1\}$ since it contains U. But G is a simple Lie group, therefore $H = G$, which proves our contention. Any curve C_y is a rational normal curve in a subspace W'_y which is necessarily G-stable. This provides infinitely many G-invariant subspaces and concludes the proof.

REMARK. Let us compare the orders of the steps in the proofs of Cartan and of Fano. In 12.4 and 12.5 the first item of business is to show that the action of h on a certain h-stable two-dimensional subspace is diagonalisable. That space is E^e in 12.4, and subsequently shown to be E^e in 12.5. Once a new eigenvector of h annihilated by e is found, 12.3 can be used. In Fano, the first step is to show that E^e is two-dimensional or rather, equivalently, that $\mathbf{P}(E)^e$ is a projective line. There, the analogue of the first step of Cartan would be to prove the existence of *two* fixed points on $\mathbf{P}(E)^e$ of h, or of the group $H = \{e^{th}\}$ generated by h. One is W^e. In the generic case $m > n$, Fano's argument may also be viewed as a search for this second fixed point: it is the intersection of $\mathbf{P}(E)^e$ with the (unique) directrix C_B. However, since the proof provides directly the G-orbit C_B of that second fixed point, the argument is not phrased in that way.

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