## §2. Main arguments

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Define $m(p)$ as the minimal positive integer $m$ such that $p^{m}>m 2^{p}$. We have $m(p) \sim p \log 2 / \log p$. In §3.3, we shall show in a simple way that $d(p) \leq 2 m(p)$ (perhaps an essentially optimal bound). Proving good lower bounds for $d(p)$ is more difficult. With the help of (1) it is easy to show that $d(p)>\sqrt{p}$. This is essentially the best that we can extract from (1). In fact, we have already remarked that (1) does not provide any information for $d>3+\sqrt{p}$. Here we give a short elementary proof of the following

THEOREM. We have $d^{2}(p)+3 d(p) \geq 2 p+2$, hence $d(p) \geq \sqrt{2 p}-\frac{3}{2}$.
An immediate corollary is that the number of solutions in $\mathbf{F}_{p}^{2}$ of $y^{2}=f(x)$ with $y \neq 0$, is at least $\sqrt{2 p}-\frac{3}{2}-d$, provided $f \in \mathbf{F}_{p}[X]$ has degree $d$ and at least one simple root. In fact, let

$$
S:=\left\{u \in \mathbf{F}_{p}: f(u) \text { is a nonzero square in } \mathbf{F}_{p}\right\}
$$

and put $g(X):=\prod_{u \in S}(X-u)$. Then observe that if $a$ is a quadratic nonresidue $\bmod p$, the polynomial $g(X)^{2} a f(X)$ assumes only square values on $\mathbf{F}_{p}$, without being a square. The theorem implies $2 \operatorname{deg} g+d \geq \sqrt{2 p}-\frac{3}{2}$. On the other hand, $2 \operatorname{deg} g$ is precisely the number of solutions we are considering. We shall outline in $\S 3.2$ how to improve on this bound.

## §2. MAIN ARGUMENTS

We start with a simple example to outline the origin of the method. We give a self-contained nine-line proof of the following claim: Let $q=2 r+1>3$ be an odd prime power and let $f \in \mathbf{F}_{q}[X]$ be a cubic polynomial. Then the equation $y^{2}=f(x)$ has at least one solution $\left(x_{0}, y_{0}\right) \in \mathbf{F}_{q}^{2}$.
(Mordell [Mo, p. 41] had to invoke fairly complicated arguments even to deal with the special case $f(X)=X^{3}+k$.)

Assume the assertion false. Then $f(u)^{r}=-1$ for all $u \in \mathbf{F}_{q}$. Hence every element of $\mathbf{F}_{q}$ is a root of $f(X)^{r}+1$ and so, identically,

$$
\begin{equation*}
f(X)^{r}+1=\left(X^{q}-X\right) S(X), \tag{2}
\end{equation*}
$$

where $S \in \mathbf{F}_{q}[X]$ has degree $3 r-q=r-1$. Differentiating the equation we get

$$
\begin{equation*}
r f^{\prime}(X) f(X)^{r-1}=\left(X^{q}-X\right) S^{\prime}(X)-S(X) \tag{3}
\end{equation*}
$$

Multiply (2) by $r f^{\prime}(X),(3)$ by $f(X)$ and subtract to obtain

$$
\begin{equation*}
r f^{\prime}(X)=\left(X^{q}-X\right)\left(r f^{\prime}(X) S(X)-f(X) S^{\prime}(X)\right)+f(X) S(X) \tag{4}
\end{equation*}
$$

Observe now that $r f^{\prime}(X)-f(X) S(X)$ has degree $3+\operatorname{deg} S=r+2$ and is divisible by $X^{q}-X$, in view of (4). Hence $r+2 \geq q=2 r+1$, i.e. $r \leq 1$ and $q \leq 3$.

We now prove the theorem. Suppose that $f \in \mathbf{F}_{p}[X](p>3)$ has degree $d \leq p-3$, is not a square in $\mathbf{F}_{p}[X]$ but assumes on $\mathbf{F}_{p}$ only values which are squares in $\mathbf{F}_{p}$. Write $f(X)=a \prod_{i=1}^{h} f_{i}(X)^{m_{i}}$, where $a \in \mathbf{F}_{p}^{*}$, the $f_{i} \in \mathbf{F}_{p}[X]$ are distinct monic irreducible polynomials and the $m_{i}$ are positive integers. Factoring out suitable even powers of the $f_{i}$, we may assume ${ }^{2}$ ) that $1 \leq m_{i} \leq 2$. Since $d<p$, there exists $u \in \mathbf{F}_{p}$ with $f(u) \neq 0$, so $f(u)$ is a nonzero square in $\mathbf{F}_{p}$. If all the $m_{i}$ were even, then $a$ would be a nonzero square in $\mathbf{F}_{p}$ and $f$ would be a square in $\mathbf{F}_{p}[X]$, contrary to assumptions. Therefore at least one of the $m_{i}$ is equal to 1 , proving that $f$ has at least a simple root $\alpha$ (in some finite field).

Let now $u \in \mathbf{F}_{p}$. Then, writing $p=2 r+1$, either $f(u)=0$ or $f(u)^{r}=1$. Therefore $f(X)\left(f(X)^{r}-1\right)$ is divisible by $X^{p}-X$. We write

$$
\begin{equation*}
f(X)^{r+1}-f(X)=\left(X^{p}-X\right) S(X), \tag{5}
\end{equation*}
$$

where $S \in \mathbf{F}_{p}[X]$ has degree $(r+1) d-p$. Differentiate (5) to obtain

$$
\begin{equation*}
(r+1) f^{\prime}(X) f^{r}(X)-f^{\prime}(X)=\left(X^{p}-X\right) S^{\prime}(X)-S(X) . \tag{6}
\end{equation*}
$$

Similarly to the above example, multiply (5) by $(r+1) f^{\prime}(X)$, (6) by $f(X)$ and subtract. The result is

$$
\begin{equation*}
f(X) S(X)=\left(X^{p}-X\right)\left(f(X) S^{\prime}(X)-(r+1) f^{\prime}(X) S(X)\right)-r f(X) f^{\prime}(X) . \tag{7}
\end{equation*}
$$

This equation is the first step in a recursion that we are going to construct. Define the differential operators $\Delta_{m}$ on $\mathbf{F}_{p}[X]$ by setting, for $\phi \in \mathbf{F}_{p}[X]$,

$$
\Delta_{m}(\phi)(X):=f(X) \phi^{\prime}(X)-(r+m+1) f^{\prime}(X) \phi(X),
$$

and put, for $m \geq 0$,

$$
\begin{cases}S_{0}(X):=S(X), & S_{m+1}(X):=\Delta_{m}\left(S_{m}\right)(X),  \tag{8}\\ R_{0}(X):=-r f(X) f^{\prime}(X), & R_{m+1}(X):=\Delta_{m+1}\left(R_{m}\right)(X) .\end{cases}
$$

Then (7) reads

$$
\begin{equation*}
f(X) S_{0}(X)=\left(X^{p}-X\right) S_{1}(X)+R_{0}(X) . \tag{9}
\end{equation*}
$$

[^0]We shall prove by induction that for all $m \geq 0$ we have

$$
\begin{equation*}
(m+1) f(X) S_{m}(X)=\left(X^{p}-X\right) S_{m+1}(X)+R_{m}(X) . \tag{10}
\end{equation*}
$$

For $m=0$ this is just (9). Assume (10) true and apply to both sides the operator $\Delta_{m}$. Note that $\Delta_{m}(\phi \psi)=\phi \Delta_{m}(\psi)+\phi^{\prime} f \psi$. We obtain

$$
(m+1) f \Delta_{m}\left(S_{m}\right)+(m+1) f^{\prime} f S_{m}=\left(X^{p}-X\right) \Delta_{m}\left(S_{m+1}\right)-f S_{m+1}+\Delta_{m}\left(R_{m}\right)
$$

Now use (10) to substitute for $(m+1) f S_{m}$ in the second term of the left side. We get
$(m+1) f S_{m+1}+f^{\prime}\left(\left(X^{p}-X\right) S_{m+1}+R_{m}\right)=\left(X^{p}-X\right) \Delta_{m}\left(S_{m+1}\right)-f S_{m+1}+\Delta_{m}\left(R_{m}\right)$, whence

$$
(m+2) f S_{m+1}=\left(X^{p}-X\right)\left(\Delta_{m}\left(S_{m+1}\right)-f^{\prime} S_{m+1}\right)+\Delta_{m}\left(R_{m}\right)-f^{\prime} R_{m}
$$

Now, to conclude the inductive argument we have only to note that $\Delta_{m}(\phi)-f^{\prime} \phi$ equals just $\Delta_{m+1}(\phi)$.

Recall that $f$ has a simple root $\alpha$. We continue by proving the following

CLAIM. Let $m \leq r$. Then $\alpha$ cannot be a double root of $S_{m}$. In particular, $S_{m}(X) \neq 0$ for $m \leq r$.

For $m=0$ this follows at once from (5). Suppose the claim true for a certain $m$ and assume by contradiction that $\alpha$ is a double root of $S_{m+1}(X)=f(X) S_{m}{ }^{\prime}(X)-(r+m+1) f^{\prime}(X) S_{m}(X)$, where $m+1 \leq r$. Then, first of all we would have $(r+m+1) f^{\prime}(\alpha) S_{m}(\alpha)=0$. This implies that $S_{m}(\alpha)=0$, since $f^{\prime}(\alpha) \neq 0$ and since $r+m+1 \leq 2 r=p-1$. Next, we compute

$$
\begin{aligned}
& S_{m+1}{ }^{\prime}(X)=f^{\prime}(X) S_{m}{ }^{\prime}(X)+f(X) S_{m}{ }^{\prime \prime}(X) \\
& \quad-(r+m+1) f^{\prime \prime}(X) S_{m}(X)-(r+m+1) f^{\prime}(X) S_{m}{ }^{\prime}(X) .
\end{aligned}
$$

Since $f(\alpha)=S_{m}(\alpha)=S_{m+1}{ }^{\prime}(\alpha)=0$, we obtain that $-(r+m) f^{\prime}(\alpha) S_{m}{ }^{\prime}(\alpha)=0$. As before, this implies that $S_{m}{ }^{\prime}(\alpha)=0$. Hence $\alpha$ would be a double root of $S_{m}(X)$, a contradiction to the inductive assumption.

As in the example, we shall conclude by comparison of degrees. Define

$$
\rho_{m}:=\operatorname{deg} R_{m}, \quad \sigma_{m}:=\operatorname{deg} S_{m},
$$

where we may agree that the zero polynomial has degree $-\infty$. We have $\rho_{0}=2 d-1$ and we derive directly from the recursion formulae (8) that $\rho_{m+1} \leq \rho_{m}+d-1$, whence

$$
\begin{equation*}
\rho_{m} \leq d+(m+1)(d-1) . \tag{11}
\end{equation*}
$$

Also, from (5), (10) and (11) we get (recalling our definition of $\operatorname{deg} 0$ ),

$$
\left\{\begin{array}{l}
\sigma_{0}=(r+1) d-p  \tag{12}\\
\sigma_{m+1} \leq \max \left(\sigma_{m}+d, \rho_{m}\right)-p \leq \max \left(\sigma_{m},(m+1)(d-1)\right)+d-p
\end{array}\right.
$$

Observe that we have $\sigma_{0}=(r+1) d-p=(r+1) d-(2 r+1)=$ $(d-2) r+(d-1) \geq d-1$. Suppose that the inequality

$$
\begin{equation*}
\sigma_{m} \geq(m+1)(d-1) \tag{13}
\end{equation*}
$$

is true for $m=0, \ldots, M-1$, but not for $m=M$ (possibly $M=\infty$ ). Then $M \geq 1$. Moreover, by (12) we have $\sigma_{m+1} \leq \sigma_{m}+d-p$ for $m \leq M-1$, whence

$$
\begin{equation*}
\sigma_{m} \leq \sigma_{0}+m(d-p)=r d-(m+1)(p-d), \quad \text { for } m \leq M \tag{14}
\end{equation*}
$$

Applying (13) and (14) with any $m \leq M-1$, we get $r d-(m+1)(p-d) \geq$ $(m+1)(d-1)$, i.e. $2 r(m+1) \leq r d$. Therefore we have

$$
\begin{equation*}
M \leq \frac{d}{2} . \tag{15}
\end{equation*}
$$

Finally, apply (12) for $m=M$ and observe that $M \leq d / 2 \leq r-1$, hence $S_{M+1} \neq 0$ by the Claim. We obtain $0 \leq \sigma_{M+1} \leq(M+1)(d-1)+d-p$, whence, comparing with (15),

$$
2 p \leq \begin{cases}d^{2}+3 d-2 & \text { if } d \text { is even } \\ d^{2}+2 d-1 & \text { if } d \text { is odd }\end{cases}
$$

This proves the theorem and more.

## §3. REMARKS

(1) The method gives some information also in the case of a general finite field $\mathbf{F}_{q}$. The same arguments as above work everywhere, on replacing $p$ by $q$, except that in the Claim we must now suppose that $m \leq r_{0}$, where $p=2 r_{0}+1$. The final conclusion will be that $d \geq \min \left(r_{0}, \sqrt{2 q}-(3 / 2)\right)$. This is still sufficient to prove that equations $y^{2}=f(x)$ in $\mathbf{F}_{q}$ have some solution, provided $p$ is sufficiently large compared to $\operatorname{deg} f$.
(2) The same method of proof produces a lower bound for the number $N^{\prime}$ of solutions of $y^{2}=f(x)$ such that $y \neq 0$. This bound is better than the one which has been stated above, as a corollary of the theorem itself. To


[^0]:    ${ }^{2}$ ) Note that when $m_{i}$ is even we cannot factor out $f_{i}(X)^{m_{i}}$ without danger of destroying the properties of $f(X)$. In fact we could have a priori $f(u)=f_{i}(u)=0$ for some $u \in \mathbf{F}_{p}$ while $\left(f / f_{i}^{m_{i}}\right)(u)$ could be a non-square in $\mathbf{F}_{p}$. It is however safe to factor out $f_{i}^{m_{i}-2}$.

