4. The three gap theorem

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• The three distance theorem is a geometric illustration of the properties of good approximation of the n-Farey points. Indeed, the two intervals containing 0 are of distinct lengths and their lengths are the two smallest. We thus have

$$\alpha q^{(1)} - p^{(1)} = \inf \{k\alpha, \text{ for } 0 \leq k \leq n\}$$

and

$$p^{(2)} - \alpha q^{(2)} = 1 - \sup \{k\alpha, \text{ for } 0 \leq k \leq n\}.$$

• For a deeper study of the rational case, the reader is referred for instance to [51].

4. THE THREE GAP THEOREM

The following theorem, called the *three gap theorem*, is in some sense the dual of the three distance theorem. This theorem was first proved by Slater (see [49] and see also [50, 51]), in the early fifties, whereas the first proofs of the three distance theorem date back to the late fifties. For other proofs of the three-gap theorem, see also [25], and more recently, [58] and [35].

The formulation of the three gap theorem quoted below is due to Slater. Following the notation of [51], let k_i be the sequence of integers k satisfying $k\alpha < \beta$. Then any difference $k_{i+1} - k_i$ is called a *gap*. Moreover, the *frequency* of a gap is defined as its frequency in the sequence of the successive gaps $(k_{i+1} - k_i)_{i \in \mathbf{N}}$.

THREE GAP THEOREM. *Let α be an irrational number in $]0,1[$ and let $\beta \in]0, 1/2[$. The gaps between the successive integers j such that $\{\alpha j\} < \beta$ take at most three values, one being the sum of the other two.*

More precisely, let $\left(\frac{p_k}{q_k}\right)_{k \in \mathbf{N}}$ and $(c_k)_{k \in \mathbf{N}}$ be the sequences of the convergents and partial quotients associated to α in its continued fraction expansion. Let $\eta_k = (-1)^k (q_k \alpha - p_k)$. There exists a unique expression for β of the form

$$\beta = m\eta_k + \eta_{k+1} + \psi,$$

with $k \geq 0$, $0 < \psi \leq \eta_k$, and if $k = 0$ then $1 \leq m \leq c_1 - 1$; otherwise, $1 \leq m \leq c_{k+1}$. Then the gaps between two successive j such that $\{j\alpha\} \in [0, \beta[$ satisfy the following:

- *the gap q_k has frequency $(m-1)\eta_k + \eta_{k+1} + \psi$,*
- *the gap $q_{k+1} - mq_k$ has frequency ψ,*
- *the gap $q_{k+1} - (m-1)q_k$ has frequency $\eta_k - \psi$.*

REMARKS.

• Suppose that α is an irrational number. By density of the sequence $(\{n\alpha\})_{n\in\mathbf{N}}$, this theorem still holds when considering the gaps between the successive integers k such that $\{\alpha k\} \in I$, where I denotes any interval of the unit circle of length β.

• Furthermore, the third gap, which is the largest, can have frequency 0, when $\eta_k = \psi$, with the above notation. This means that this gap does not appear at all, as a consequence of the uniform distribution of the sequence $(\{n\alpha\})_{n\in\mathbf{N}}$ in the circle.

• The other two gaps do always appear (infinitely often, in fact, because of their positive frequencies) and are shown to be equal to the smallest positive integers l_1 and l_2 such that $\{l_1\alpha\} < \beta$ and $\{l_2\alpha\} > 1-\beta$ (see [51]).

• The study of the rational case proves the equivalence between the three distance and the three gap theorems, as observed by Slater [51] in the case of an open interval and by Langevin, for any interval, in [35].

4.1 CONNECTEDNESS INDEX

Let $u = (u_n)_{n\in\mathbf{N}}$ be a coding of a rotation by irrational angle $0 < \alpha < 1$ with respect to the partition

$$\mathcal{P} = \{[\beta_0, \beta_1[, [\beta_1, \beta_2[, \ldots, [\beta_{p-1}, \beta_p[\} .$$

We have seen in Section 2.1 that the sets $I(w_1, \ldots, w_n) = \bigcap_{j=0}^{n-1} R^{-j}(I_{w_{j+1}})$, where $I_k = [\beta_k, \beta_{k+1}[$, for $0 \le j \le p-1$, are connected except for $w_1 \cdots w_n$ of the form a_K^n, where K denotes the index of the interval of $\mathcal{P}$ (if such an interval exists) of length greater than $\sup(\alpha, 1-\alpha)$.

Let us suppose that there exists an interval of $\mathcal{P}$ of length L greater than $1-\alpha$ and index K, say. We deduce from the three gap theorem that the set of integers n such that a_K^n is a factor of the sequence u is bounded. More precisely, let us define $n^{(1)}$ as the largest integer n such that a_K^n is a factor of the sequence u. We will call the integer $n^{(1)}$ the *index of connectedness* of the sequence u. (If every interval of $\mathcal{P}$ has length smaller than or equal to $\sup(\alpha, 1-\alpha)$ then the connectedness index of u is equal to 1.) The three gap theorem enables us to give an exact expression for the connectedness index. Indeed $n^{(1)} + 1$ is the largest gap between the consecutive values of k for which $0 < \{k\alpha\} < 1-L$. We thus have the following

THEOREM 9. *Let $u = (u_n)_{n\in\mathbf{N}}$ be a coding of the rotation by irrational angle α. Suppose that there exists an interval of $\mathcal{P}$ of length $L > \sup(\alpha, 1-\alpha)$. Let $\left(\frac{p_k}{q_k}\right)_{k\in\mathbf{N}}$ and $(c_k)_{k\in\mathbf{N}}$ be the sequences of convergents and partial quotients associated to α in its continued fraction expansion. Let $\eta_k = (-1)^k(q_k\alpha - p_k)$. Write*

$$1 - L = m\eta_k + \eta_{k+1} + \psi\,,$$

with $k \geq 1$, $0 < \psi \leq \eta_k$ and $1 \leq m \leq c_{k+1}$. The connectedness index $n^{(1)}$ of the sequence u satisfies

$$n^{(1)} = q_{k+1} - (m-1)q_k - 1,\ \text{if } \psi \neq \eta_k\,,$$
$$n^{(1)} = q_{k+1} - mq_k - 1,\ \text{if } \psi = \eta_k \text{ and } m < c_{k+1}\,,$$
$$n^{(1)} = q_k - 1,\ \text{if } \psi = \eta_k \text{ and } m = c_{k+1}\,.$$

4.2 APPLICATIONS

Precise knowledge of the connectedness index is useful, as shown by the following. Indeed Lemma 1 can be rephrased as follows.

LEMMA 3. *Let u be a coding of an irrational rotation on the unit circle with respect to the partition $\{[\beta_0, \beta_1[,\ [\beta_1, \beta_2[, \ldots, [\beta_{p-1}, \beta_p[\}$. The frequencies of factors of u of length $n \geq n^{(1)}$, where $n^{(1)}$ denotes the connectedness index, are equal to the lengths of the intervals bounded by the points*

$$\{k(1-\alpha) + \beta_i\},\ \text{for } 0 \leq k \leq n-1,\ \ 0 \leq i \leq p-1\,.$$

The complexity of a coding on p letters of an irrational rotation ultimately has the form $p(n) = an + b$, where $a \leq p$, and depends on the algebraic relations between the angle and the lengths of the intervals of the coding. More precisely, we have the following theorem proved in [1].

THEOREM 10. *Let $u = (u_n)_{n\in\mathbf{N}}$ be a coding of the irrational rotation R of irrational angle α with respect to the partition*

$$\mathcal{P} = \{[\beta_0, \beta_1[,\ [\beta_1, \beta_2[, \ldots, [\beta_{p-1}, \beta_p[\}\,.$$

Let $(k_n)_{n\in\mathbf{N}}$ be the sequence defined by

$$k_0 = p = \operatorname{card}(F)\,,$$

$$k_n = \operatorname{card}\left\{\beta_i \in F;\ \forall k \in [1, \ldots, n],\ R^{-k}(\beta_i) \notin F\right\}\,.$$

Let a be the limit of this sequence, $n^{(2)}$ the smallest index such that $k_n = a$, and let

$$b = \sum_{i=0}^{n^{(2)}-1} (k_i - a) .$$

Let $n^{(1)}$ denote the connectedness index of u.

If $n \geq \max(n^{(1)}, n^{(2)})$, then the complexity of the sequence u satisfies

$$p(n) = an + b .$$

REMARKS.

• Note that if $1, \alpha, \beta_1, \ldots, \beta_p$ are rationally independent, then $n^{(2)} = 0$, $b = 0$ and $a = p$.

• Theorem 10 answers the question of the existence of sequences of ultimately affine complexity (for more details, the reader is referred to [1], see also the result of Cassaigne in [11]).

4.3 BEATTY SEQUENCES

The connections between the three gap theorem and the Beatty sequences have been investigated by Fraenkel and Holzman in [26]. Let us recall that a Beatty sequence is a sequence $u(\alpha, \rho) = (u_n)_{n\in\mathbf{N}}$ of the form $u_n = \lfloor \alpha n + \rho \rfloor$, where α and ρ are real numbers such that $\alpha \geq 1$. The number α is called the *modulus* and ρ is called the *residue* or *intercept*. For an impressive bibliography on the subject, we refer the reader to [27] and [54]. Fraenkel and Holzman have noticed in [26] that the three gap theorem answers the question of the gaps in the intersection of a Beatty sequence and an arithmetical sequence $(an + c)_{n\in\mathbf{N}}$, for a a positive integer and c an integer. This result has been obtained independently by Wolff and Pitman in [58]. By intersection of the two Beatty sequences $s = (s_n)_{n\in\mathbf{N}}$ and $t = (t_n)_{n\in\mathbf{N}}$, we mean the strictly increasing sequence u defined as:

$$\{u_n,\ n \in \mathbf{N}\} = \{u,\ \exists k, l \in \mathbf{N} \text{ such that } u = s_k = t_l\} .$$

Hence a gap in the intersection denotes the difference between two distinct elements of the intersection.

Note that Beatty sequences and Sturmian sequences are related: let u be a Beatty sequence of modulus α and residue ρ; the characteristic sequence $(v_n)_{n\in\mathbf{N}}$ of u defined as

$$v_n = 1 \text{ if and only if there exists } m \text{ such that } n = \lfloor \alpha m + \rho \rfloor$$

is the Sturmian sequence obtained as the coding of the orbit of $-\rho/\alpha$ under the rotation by angle $1/\alpha$, with respect to the partition

$$\{]0, 1-1/\alpha],]1-1/\alpha, 1]\} .$$

Indeed, if $n = \lfloor \alpha m + \rho \rfloor$, then $\lceil 1/\alpha(n+1) - \rho/\alpha \rceil = m+1 = 1 + \lceil n/\alpha - \rho/\alpha \rceil$, and if $\lfloor \alpha m + \rho \rfloor < n < \lfloor \alpha(m+1) + \rho \rfloor$, then $\lceil 1/\alpha(n+1) - \rho/\alpha \rceil = \lceil n/\alpha - \rho/\alpha \rceil$.

5. THE RECURRENCE FUNCTION

Let us deduce now from the three distance and three gap theorems a simple proof of the following result originally due to Morse and Hedlund concerning the recurrence function of a Sturmian sequence (see [40]).

Recall that a sequence u is called *minimal* or *uniformly recurrent* if every factor of u appears infinitely often and with bounded gaps or, equivalently, if for any integer n, there exists an integer m such that every factor of u of length m contains every factor of u of length n. Note that it is equivalent (see [30]) to the *minimality* of the dynamical system $(\overline{\mathcal{O}(u)}, T)$, i.e., the orbit of every element of $\overline{\mathcal{O}(u)}$ is dense, or equivalently every sequence in the orbit closure of u has the same set of factors as u.

The recurrence function φ of a minimal sequence u is defined by

$$\varphi(n) = \min\{m \in \mathbf{N} \text{ such that } \forall B \in L_m, \ \forall A \in L_n, \ A \text{ is a factor of } B\} ,$$

where L_n denotes the set of factors of u of length n, i.e., $\varphi(n)$ is the size of the smallest window that contains all factors of length n whatever its position in the sequence.

THEOREM 11. *Let u be a Sturmian sequence with angle α. Let $(q_k)_{k\in\mathbf{N}}$ denote the sequence of denominators of the convergents of the continued fraction expansion of α. The recurrence function φ of this sequence satisfies for any non zero integer n :*

$$\varphi(n) = n - 1 + q_k + q_{k-1}, \quad \textit{where} \quad q_{k-1} \leq n < q_k .$$

Proof of Theorem 11. Let $u \in \{0,1\}^{\mathbf{N}}$ be a Sturmian sequence. There exist a real number x and an irrational number α in $]0,1[$ such that $u_n = 0 \Leftrightarrow \{x + n\alpha\} \in I_0$, with $I_0 = [0, \alpha[$ or $I_0 =]0, \alpha]$ (see Section 2.1). Let $I_1 = [\alpha, 1[$ (respectively, $]\alpha, 1]$) if $I_0 = [0, \alpha[$ (respectively, $I_0 =]0, \alpha]$). Let us denote by R the rotation of the circle by angle α. Assume we are given