6. HIGHER-DIMENSIONAL GENERALIZATIONS

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **44 (1998)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **21.07.2024**

the form $I(w_1, \ldots, w_n)$ of length δ_n; therefore, there exists a factor of u of length $l_n - 2 + n$ which does not contain the factor $w_1 \cdots w_n$. This shows that $\varphi(n) \geq n - 1 + l_n$. The lemma is thus proved.

REMARK. Note that in the case of the Fibonacci sequence ($\alpha = \frac{\sqrt{5}-1}{2}$), the recurrence function satisfies, for $F_{k-1} < n \leq F_k$,

$$\varphi(n) = n - 1 + F_{k+1},$$

where $(F_n)_{n \in \mathbf{N}}$ denotes the Fibonacci sequence $F_{n+1} = F_n + F_{n-1}$, with $F_0 = 1$ and $F_1 = 2$.

This result is extended in [13] to the fixed point of the substitution σ introduced by Rauzy which generalizes the Fibonacci substitution and is defined by $\sigma(0) = 01$, $\sigma(1) = 02$, $\sigma(2) = 0$.

THEOREM 12. *Let T_n denote the so-called Tribonacci sequence defined as follows: $T_{k+3} = T_{k+2} + T_{k+1} + T_k$, with $T_0 = 0$, $T_1 = 0$, $T_1 = 1$. The recurrence function φ of the fixed point beginning with 0 of the Rauzy substitution satisfies for any positive integer n:*

$$\varphi(n) = n - 1 + T_{k+6}, \quad \textit{where} \quad \sum_{0}^{k+1} T_i < n \leq \sum_{0}^{k+2} T_i .$$

6. HIGHER-DIMENSIONAL GENERALIZATIONS

6.1 TWO-DIMENSIONAL GENERALIZATIONS AND BEATTY SEQUENCES

Let us consider now some two-dimensional versions of the three distance and three gap theorems. Such generalizations were introduced by Fraenkel and Holzman in [26] in order to give an upper bound for the number of gaps in the intersection of two Beatty sequences. They first reduce this problem to a two-dimensional version of the three distance theorem, conjectured by Simpson and Holzman and proved by Geelen and Simpson (see [29]). Then they deduce from this theorem a bound for the number of gaps in the intersection of two Beatty sequences, when at least one of the moduli is rational.

Let us first give the two-dimensional version of the three gap theorem introduced by Fraenkel and Holzman. We will use the same notation as in [26]: for any pair of real numbers (x, y), $\{(x, y)\}$ means the equivalence class of (x, y) mod $\mathbf{Z}^2$, i.e., $\{(x, y)\}$ belongs to the torus $\mathbf{T}^2$.

THEOREM 13. *Let* α_1, α_2, β_1, β_2, μ_1 *and* μ_2 *be real numbers in* $[0,1[$. *The gaps between the successive values of the integers* n *such that the following points of the torus* $\mathbf{T}^2$

$$\{(n\alpha_1,\, n\alpha_2)\}$$

belong to the rectangle

$$\mathcal{R} = \{\{(x,y)\}\,;\ \mu_1 - \beta_1 < x \leq \mu_1,\ \mu_2 - \beta_2 < y \leq \mu_2\}$$

take a finite number of values which depend only on α_1, α_2, β_1 *and* β_2.

Furthermore, if at least one of the two angles α_1 *and* α_2 *is rational, then the number of gaps is bounded by* $q+3$, *where* q *is the minimum of the denominators of* α_1 *and* α_2 *in lowest terms (the denominator of an irrational number is considered as* $+\infty$*).*

Let us state now the two-dimensional version of the three distance theorem proved in [29] by Geelen and Simpson.

THEOREM 14. *Assume we are given two real numbers* α_1, α_2 *and two positive integers* n_1, n_2. *The set of points*

$$\{i\alpha_1 + j\alpha_2 + \rho,\ 0 \leq i \leq n_1 - 1,\ 0 \leq j \leq n_2 - 1\}$$

partitions the unit circle into intervals having at most $\min\{n_1, n_2\} + 3$ *lengths.*

Note that the bound $\min\{n_1, n_2\} + 3$ is not the best possible when n_1 or $n_2 = 1$. Indeed, in this case, the statement reduces to the three distance theorem. For a discussion on the achievability of the bound, the reader is referred to [29].

Fraenkel and Holzman have proved in [26] that Theorems 13 and 14 together answer the question of the intersection of two Beatty sequences, when at least one modulus is rational. We define a *gap* in the intersection of two Beatty sequences to be a difference between two successive elements of the intersection, and an *index-gap* to be the difference between the two corresponding indices in the same Beatty sequence.

THEOREM 15. *Let* $(\lfloor n\alpha_1 + \rho_1 \rfloor)_{n\in\mathbf{N}}$ *and* $(\lfloor n\alpha_2 + \rho_2 \rfloor)_{n\in\mathbf{N}}$ *be two Beatty sequences, with at least one of the two moduli* α_1 *and* α_2 *rational. Let* q *denote the minimum of the denominators of* α_1 *and* α_2 *in lowest terms (the denominator of an irrational number is considered as* $+\infty$*). The number of gaps and index-gaps in the intersection is bounded by* $q+3$, *if* $q \geq 2$, *and bounded by* 3 *otherwise.*

Fraenkel and Holzman show furthermore that this bound is achievable and that the number of gaps can be made arbitrarily large, when at least one of the moduli is rational.

6.2 Combinatorial applications

Now let us review some applications of Theorems 13 and 14. For instance we can deduce the following result for the intersection of two Sturmian sequences.

THEOREM 16. *Let $s = (s_n)_{n \in \mathbf{N}}$ and $t = (t_n)_{n \in \mathbf{N}}$ be two Sturmian sequences. The number of gaps between the successive integers n such that $s_n = t_n$ is finite.*

Proof. Let $s = (s_n)_{n \in \mathbf{N}}$ and $t = (t_n)_{n \in \mathbf{N}}$ be two Sturmian sequences of angles α and β, with corresponding partitions $\{I_0, I_1\}$ and $\{J_0, J_1\}$. The gaps between the integers n such that the points $\{(n\alpha, n\beta)\}$ in $\mathbf{T}^2$ belong to the rectangle $I_0 \times J_0$ (respectively, $I_1 \times J_1$) take a finite number of values, hence so do the gaps between the successive integers n such that the points $\{(n\alpha, n\beta)\}$ in $\mathbf{T}^2$ belong to the set $I_0 \times J_0 \cup I_1 \times J_1$.

We also deduce from Theorem 14 and Lemma 3 the following

THEOREM 17. *Let u be a coding of the irrational rotation by angle $0 < \alpha < 1$ with respect to a partition into d intervals of length $1/d$. The frequencies of factors of u of length $n \geq \sup\{n^{(1)}, d\}$ take at most $d+3$ values, where $n^{(1)}$ denotes the connectedness index.*

Proof. This result is a direct application of Lemma 3 and Theorem 14. Indeed, the intervals $I(w_1, \ldots, w_n)$ (corresponding to the factors $w_1 \cdots w_n$ of length n) are bounded by the points

$$\{i(1-\alpha) + j/d, \quad 0 \leq i \leq n-1, \quad 0 \leq j \leq d-1\} .$$

Vuillon has introduced in [57] two-dimensional generalizations of Sturmian sequences obtained by considering the approximation of a plane of irrational normal by square faces oriented along the three coordinates planes. Theorem 14 can also be applied to give an upper bound for the number of frequencies of blocks of a given size for such double sequences (see [4]).

We will give in Section 7 a direct combinatorial proof of Theorem 14 in the particular case $\min\{n_1, n_2\} = 2$, and give an interpretation in terms of frequencies of binary codings: the frequencies of the factors of given length of a coding of an irrational rotation with respect to a partition in two intervals take ultimately at most 5 values.

6.3 THE $3d$ DISTANCE THEOREM

Let us consider another generalization of the three distance theorem, known as the $3d$ *distance theorem.* This result, conjectured by Graham (see [17] and [34]), was first proved by Chung and Graham in [18] and secondly by Liang who gave a very nice proof in [37]. Geelen and Simpson remark in [29] that their proof uses ideas from Liang's proof.

THE $3d$ DISTANCE THEOREM. *Assume we are given $0 < \alpha < 1$ irrational, $\gamma_1, \ldots, \gamma_d$ real numbers and $n_1, \ldots, n_d$ positive integers. The points $\{n\alpha + \gamma_i\}$, for $0 \leq n < n_i$ and $1 \leq i \leq d$, partition the unit circle into at most $n_1 + \cdots + n_d$ intervals, having at most $3d$ different lengths.*

We will give a combinatorial proof of this result in Section 8 and express the corresponding result for frequencies of codings of rotations, i.e., that the frequencies of the factors of given length of a coding of a rotation by the unit circle under a partition in d intervals take ultimately at most $3d$ values.

6.4 OTHER GENERALIZATIONS

Slater has studied in [50] the following generalization of the three gap theorem, which should be compared with Theorem 13: there is a bounded number of gaps between the successive values of the integers n such that $\{n(\eta_1, \ldots, \eta_d)\} \in C$, where C is a closed convex region on the d-dimensional torus and where $1, \eta_1 \ldots, \eta_d$ are rationally independent. However, Fraenkel and Holzman prove Theorem 13 even in the case where α_1, α_2 and 1 are rationally independent.

Chevallier studies in [16] a d-generalization of the three distance theorem to $\mathbf{T}^d$, where intervals are replaced by Voronoï cells: the number of Voronoï cells (up to isometries) is shown to be connected to the number of sides of a Voronoï cell. The notion of continued fraction expansion is generalized by properties of best approximation.

Finally, note the unsolved problems quoted in [29] concerning further generalizations of the three distance theorem. For instance, an upper bound for the number of distinct lengths in the partition of the unit circle by the points

$k_1\alpha_1 + k_2\alpha_2 + \cdots + k_d\,\alpha_d$, for $k_i \leq n_i - 1$ and $1 \leq i \leq d$ is conjectured to be of the form $c_d + \prod_{i=1}^{d-1} n_i$, where c_d is a constant independent of $n_1, \ldots, n_d$.

7. Frequencies of factors for binary codings of rotations

We will prove in this section the following result, which corresponds to the case $\min\{n_1, n_2\} = 2$ in Theorem 14. The idea of using a reflection of the unit circle can also be found in the original proof in [29].

THEOREM 18. *Let α be an irrational number in $]0,1[$, $\beta \neq 0$ a real number and n a non-zero integer. The set of points $\{0\}, \{\beta\}, \{\alpha\}$, $\{\beta+\alpha\}, \ldots, \{n\alpha\}, \{\beta+n\alpha\}$ divides the circle into a finite number of intervals, whose lengths take at most five values.*

7.1 A combinatorial proof

We will prove Theorem 18 by introducing a coding of the rotation by angle α with respect to the intervals of the unit circle bounded by the points $\{0\}, \{\beta\}, \{\alpha\}, \{\beta+\alpha\}, \ldots, \{n\alpha\}, \{\beta+n\alpha\}$.

Let α be an irrational number, β a non-zero real number and n an integer. Let $I_1, \ldots, I_p$ denote the intervals of the unit circle bounded by the points $\{0\}, \{\beta\}, \{\alpha\}, \{\beta+\alpha\}, \ldots, \{n\alpha\}, \{\beta+n\alpha\}$. Let $u = (u_n)_{n\in\mathbf{N}}$ be the sequence defined on the alphabet $\Sigma = \{a_1, \ldots, a_p\}$ as the coding of the orbit of 0 under the rotation R of angle α under the partition $\{I_1, \ldots, I_p\}$:

$$u_n = a_k \Longleftrightarrow \{n\alpha\} \in I_k\,.$$

The frequency of the letter a_k in the sequence u is equal to the length of the interval I_k, by uniform distribution of the sequence $(\{n\alpha\})_{n\in\mathbf{N}}$. We must now prove that the frequencies of the letters of u take at most five values. Let us consider the graph Γ_1 of words of u of length 1. There is one edge from a_k to $a_{k'}$ if $I_{k'}$ is the image of I_k by the rotation R or if $I_{k'}$ contains $\{-\alpha\}$ or $\{-\alpha+\beta\}$. Therefore the graph Γ_1 contains p vertices (one for each letter) and $p+2$ edges: indeed, every vertex has only one leaving edge, except the ones associated with the intervals containing $\{-\alpha\}$ or $\{n-\alpha+\beta\}$, which have two leaving edges (if both of these points belong to the same interval I_k, then a_k has three leaving edges and all the other intervals have only one edge). In other words, we have $p(1) = p$ and $p(2) = p+2$. As in the proof of Theorem 6, this implies that there are at most 6 branches in Γ_1 : indeed, each