6.2 Combinatorial applications

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Fraenkel and Holzman show furthermore that this bound is achievable and that the number of gaps can be made arbitrarily large, when at least one of the moduli is rational.

6.2 Combinatorial applications

Now let us review some applications of Theorems 13 and 14. For instance we can deduce the following result for the intersection of two Sturmian sequences.

THEOREM 16. *Let $s = (s_n)_{n \in \mathbf{N}}$ and $t = (t_n)_{n \in \mathbf{N}}$ be two Sturmian sequences. The number of gaps between the successive integers n such that $s_n = t_n$ is finite.*

Proof. Let $s = (s_n)_{n \in \mathbf{N}}$ and $t = (t_n)_{n \in \mathbf{N}}$ be two Sturmian sequences of angles α and β, with corresponding partitions $\{I_0, I_1\}$ and $\{J_0, J_1\}$. The gaps between the integers n such that the points $\{(n\alpha, n\beta)\}$ in $\mathbf{T}^2$ belong to the rectangle $I_0 \times J_0$ (respectively, $I_1 \times J_1$) take a finite number of values, hence so do the gaps between the successive integers n such that the points $\{(n\alpha, n\beta)\}$ in $\mathbf{T}^2$ belong to the set $I_0 \times J_0 \cup I_1 \times J_1$.

We also deduce from Theorem 14 and Lemma 3 the following

THEOREM 17. *Let u be a coding of the irrational rotation by angle $0 < \alpha < 1$ with respect to a partition into d intervals of length $1/d$. The frequencies of factors of u of length $n \geq \sup\{n^{(1)}, d\}$ take at most $d+3$ values, where $n^{(1)}$ denotes the connectedness index.*

Proof. This result is a direct application of Lemma 3 and Theorem 14. Indeed, the intervals $I(w_1, \ldots, w_n)$ (corresponding to the factors $w_1 \cdots w_n$ of length n) are bounded by the points

$$\{i(1-\alpha) + j/d, \quad 0 \leq i \leq n-1, \quad 0 \leq j \leq d-1\} .$$

Vuillon has introduced in [57] two-dimensional generalizations of Sturmian sequences obtained by considering the approximation of a plane of irrational normal by square faces oriented along the three coordinates planes. Theorem 14 can also be applied to give an upper bound for the number of frequencies of blocks of a given size for such double sequences (see [4]).

We will give in Section 7 a direct combinatorial proof of Theorem 14 in the particular case $\min\{n_1, n_2\} = 2$, and give an interpretation in terms of frequencies of binary codings: the frequencies of the factors of given length of a coding of an irrational rotation with respect to a partition in two intervals take ultimately at most 5 values.

6.3 THE $3d$ DISTANCE THEOREM

Let us consider another generalization of the three distance theorem, known as the $3d$ *distance theorem.* This result, conjectured by Graham (see [17] and [34]), was first proved by Chung and Graham in [18] and secondly by Liang who gave a very nice proof in [37]. Geelen and Simpson remark in [29] that their proof uses ideas from Liang's proof.

THE $3d$ DISTANCE THEOREM. *Assume we are given* $0 < \alpha < 1$ *irrational,* $\gamma_1, \dots, \gamma_d$ *real numbers and* $n_1, \dots, n_d$ *positive integers. The points* $\{n\alpha + \gamma_i\}$, *for* $0 \le n < n_i$ *and* $1 \le i \le d$, *partition the unit circle into at most* $n_1 + \cdots + n_d$ *intervals, having at most* $3d$ *different lengths.*

We will give a combinatorial proof of this result in Section 8 and express the corresponding result for frequencies of codings of rotations, i.e., that the frequencies of the factors of given length of a coding of a rotation by the unit circle under a partition in d intervals take ultimately at most $3d$ values.

6.4 OTHER GENERALIZATIONS

Slater has studied in [50] the following generalization of the three gap theorem, which should be compared with Theorem 13: there is a bounded number of gaps between the successive values of the integers n such that $\{n(\eta_1, \dots, \eta_d)\} \in C$, where C is a closed convex region on the d-dimensional torus and where $1, \eta_1 \dots, \eta_d$ are rationally independent. However, Fraenkel and Holzman prove Theorem 13 even in the case where α_1, α_2 and 1 are rationally independent.

Chevallier studies in [16] a d-generalization of the three distance theorem to $\mathbf{T}^d$, where intervals are replaced by Voronoï cells: the number of Voronoï cells (up to isometries) is shown to be connected to the number of sides of a Voronoï cell. The notion of continued fraction expansion is generalized by properties of best approximation.

Finally, note the unsolved problems quoted in [29] concerning further generalizations of the three distance theorem. For instance, an upper bound for the number of distinct lengths in the partition of the unit circle by the points