7. Frequencies of factors for binary codings of rotations

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$k_1\alpha_1 + k_2\alpha_2 + \cdots + k_d\alpha_d$, for $k_i \leq n_i - 1$ and $1 \leq i \leq d$ is conjectured to be of the form $c_d + \prod_{i=1}^{d-1} n_i$, where c_d is a constant independent of $n_1, \ldots, n_d$.

7. Frequencies of factors for binary codings of rotations

We will prove in this section the following result, which corresponds to the case $\min\{n_1, n_2\} = 2$ in Theorem 14. The idea of using a reflection of the unit circle can also be found in the original proof in [29].

THEOREM 18. *Let* α *be an irrational number in* $]0,1[$, $\beta \neq 0$ *a real number and* n *a non-zero integer. The set of points* $\{0\}, \{\beta\}, \{\alpha\}, \{\beta+\alpha\}, \ldots, \{n\alpha\}, \{\beta+n\alpha\}$ *divides the circle into a finite number of intervals, whose lengths take at most five values.*

7.1 A combinatorial proof

We will prove Theorem 18 by introducing a coding of the rotation by angle α with respect to the intervals of the unit circle bounded by the points $\{0\}, \{\beta\}, \{\alpha\}, \{\beta+\alpha\}, \ldots, \{n\alpha\}, \{\beta+n\alpha\}$.

Let α be an irrational number, β a non-zero real number and n an integer. Let $I_1, \ldots, I_p$ denote the intervals of the unit circle bounded by the points $\{0\}, \{\beta\}, \{\alpha\}, \{\beta+\alpha\}, \ldots, \{n\alpha\}, \{\beta+n\alpha\}$. Let $u = (u_n)_{n\in\mathrm{N}}$ be the sequence defined on the alphabet $\Sigma = \{a_1, \ldots, a_p\}$ as the coding of the orbit of 0 under the rotation R of angle α under the partition $\{I_1, \ldots, I_p\}$:

$$u_n = a_k \Longleftrightarrow \{n\alpha\} \in I_k .$$

The frequency of the letter a_k in the sequence u is equal to the length of the interval I_k, by uniform distribution of the sequence $(\{n\alpha\})_{n\in\mathrm{N}}$. We must now prove that the frequencies of the letters of u take at most five values. Let us consider the graph Γ_1 of words of u of length 1. There is one edge from a_k to $a_{k'}$ if $I_{k'}$ is the image of I_k by the rotation R or if $I_{k'}$ contains $\{-\alpha\}$ or $\{-\alpha+\beta\}$. Therefore the graph Γ_1 contains p vertices (one for each letter) and $p+2$ edges: indeed, every vertex has only one leaving edge, except the ones associated with the intervals containing $\{-\alpha\}$ or $\{n-\alpha+\beta\}$, which have two leaving edges (if both of these points belong to the same interval I_k, then a_k has three leaving edges and all the other intervals have only one edge). In other words, we have $p(1) = p$ and $p(2) = p+2$. As in the proof of Theorem 6, this implies that there are at most 6 branches in Γ_1 : indeed, each branch starts with a vertex with more than one entering edge (this provides at most two branches) or just after a vertex with at least two leaving edges (at most four branches are of this kind). We deduce from this that the frequencies of the letters in u take at most 6 values. Let us prove that at least two branches of Γ_1 have the same frequency, which will complete the proof.

Let s denote the reflection of the circle defined by $s\colon x \to \{\beta + n\alpha - x\}$. This reflection leaves invariant the endpoints of the intervals $I_1, \dots, I_p$ and thus induces a permutation σ of the interiors of the intervals I_k, which can also be seen as a permutation of Σ. The length of I_k is equal to the length of $I_{\sigma(k)} = s(I_k)$. The frequency of the letter a_k is thus equal to the frequency of the letter $\sigma(a_k)$. Note that if $a_i a_j$ is a factor of u, then $\sigma(a_j)\sigma(a_i)$ is also a factor. We deduce from this that if there is an edge in Γ_1 from a_i to a_j, then there is also an edge from $\sigma(a_j)$ to $\sigma(a_i)$, or in other words, that Γ_1 is invariant by the following action of σ : the image of the vertex associated with the letter a is equal to the vertex associated with $\sigma(a)$ and the image of the edge $a \to b$ is the edge $\sigma(b) \to \sigma(a)$, i.e., each letter is replaced by its image and the direction of every edge is changed. Furthermore, the image of a branch is a branch. Let us prove that at most four branches of the graph Γ_1 are invariant by σ. Let $B = U_1 \to U_2 \to \cdots \to U_q$ be an invariant branch of the graph. We have $B = \sigma(B) = \sigma(U_q) \to \cdots \to \sigma(U_1)$. We thus get $\sigma(U_k) = U_{q+1-k}$.

- Suppose that there exists i such that $U_i = \sigma(U_i)$. Therefore the interval I_i must contain a fixed point for s. Since there are only two such fixed points, at most two branches can satisfy this property.

- Let us suppose that $U_i \neq \sigma(U_i)$ for each $1 \leq i \leq q$. We thus have q even and $\sigma(U_{q/2}) = U_{q/2+1}$. Let I (respectively, I') be the closure of the interval associated with $U_{q/2}$ (respectively, $U_{q/2+1}$). We thus get $s(I) = I'$. Furthermore, I' is the image of I by the rotation R, because of the edge $U_{q/2} \to U_{q/2} + 1$. This implies that I' contains a fixed point of the symmetry $s \circ R^{-1}$, which has at most two fixed points. Hence, at most two branches are of this kind.

We have proved that at most four basic paths can be their own image by σ. Therefore, there exist among the six branches at least two different branches, say A and B, such that $B = \sigma(A)$. Thus A and B have the same frequency, which implies that there are at most five possible frequencies for the letters of u.

7.2 APPLICATION TO BINARY CODINGS

A more natural coding of the rotation R would have been with respect to the partition $[0,\beta[,\ [\beta,1[$. The points $\{0\},\{\beta\},\{\alpha\},\{\beta+\alpha\},\ldots,\{n\alpha\}$, $\{\beta+n\alpha\}$ are the endpoints of the sets $I(w_1,\ldots,w_n)$, following the notation of Section 2. But these sets might not be connected. Thus the frequencies of factors of length n are the sums of the lengths of the connected components of the sets $I(w_1,\ldots,w_n)$. Despite this disadvantage, this coding allows us to deduce the following result from Lemma 3.

THEOREM 19. *Let u be a coding of an irrational rotation with respect to the partition into two intervals $\{[0,\beta[,\ [\beta,1[\}$, where $0<\beta<1$. Let $n^{(1)}$ denote the connectedness index of u. The frequencies of factors of given length $n \geq n^{(1)}$ of u take at most 5 values. Furthermore, the set of factors of u is stable by mirror image, i.e., if the word $a_1\cdots a_n$ is a factor of the sequence u, then $a_n\cdots a_1$ is also a factor and furthermore, both words have the same frequency.*

Proof. It remains to prove the part of this theorem concerning the stability by mirror image. Assume we are given a fixed positive integer n. Let s_n be the reflection of the circle defined by $s_n\colon x \to \{\beta-(n-1)\alpha-x\}$. We have $s_n(R^{-k}(I_j)) = R^{(-n+1+k)}(I_j)$, for $j=0,1$, following the previous notation. The image of $I(w_1,\ldots,w_n)$ by s_n is $I(w_n,\ldots,w_1)$; they thus have the same length, which gives the result.

REMARK. A study of the topology of the graph of words for a binary coding of an irrational rotation of complexity satisfying ultimately $p(n+1)-p(n)=2$ can be found in [24] or in [46].

8. THE 3d DISTANCE THEOREM

Following the idea of the above proof, let us give a combinatorial proof of the 3d *distance theorem.*

THE 3d DISTANCE THEOREM. *Assume we are given $0<\alpha<1$ irrational, $\gamma_1,\ldots,\gamma_d$ real numbers and $n_1,\ldots,n_d$ positive integers. The points $\{n\alpha+\gamma_i\}$, for $0\leq n<n_i$ and $1\leq i\leq d$, partition the unit circle into at most $n_1+\cdots+n_d$ intervals, having at most 3d different lengths.*