1. Introduction

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1. Introduction

The motion under gravity of a rigid body one of whose points is fixed is described by a Hamiltonian system on the cotangent bundle $T^*\,\mathrm{SO}(3)$ of its configuration space $\mathrm{SO}(3)$, coordinatized by Euler angles and their conjugate momenta. This system was first obtained by Lagrange around 1788 [17], the particular case of free rigid body motion being already known to Euler. After a first reduction, with respect to rotations about the vertical in space, this leads to the following two degrees of freedom Hamiltonian system on T^*S^2, also obtained by Lagrange [17, p. 232 and p. 243]:

$$\frac{dM}{dt} = M \times \Omega + \chi \times \Gamma, \qquad \frac{d\Gamma}{dt} = \Gamma \times \Omega \tag{1}$$

$$M = (M_1, M_2, M_3), \;\; \Omega = (\Omega_1, \Omega_2, \Omega_3), \;\; \Gamma = (\Gamma_1, \Gamma_2, \Gamma_3), \;\; \chi = (\chi_1, \chi_2, \chi_3).$$

Here M, Ω and Γ denote respectively the angular momentum, the angular velocity and the coordinates of the unit vector in the direction of gravity, all expressed in body-coordinates. The constant vector χ is the center of mass in body-coordinates multiplied by the mass of the body and the acceleration. We recall that $M = I\Omega$ where I is the matrix of the inertia operator and we may suppose that $I = \mathrm{diag}\,(I_1, I_2, I_3)$. The system (1) may be viewed as a two degrees of freedom Hamiltonian system on the manifold $\mathfrak{se}^*(3) \sim \mathfrak{se}(3)$ – the Lie algebra of the Euclidean group of three space $\mathrm{SE}(3) = \mathrm{SO}(3) \times \mathbf{R}^3$. Indeed, $\mathfrak{se}^*(3)$ with its usual Kostant-Kirillov-Poisson structure may be identified, via (a multiple of) the Killing form, with $\mathfrak{se}(3)$. This induces the following Lie-Poisson bracket on $\mathfrak{se}(3) \sim \mathbf{R}^3 \times \mathbf{R}^3$

$$\{M_1, M_2\} = -M_3\,, \ldots, \qquad \{M_1, \Gamma_2\} = -\Gamma_3\,, \ldots, \qquad \{\Gamma_i, \Gamma_j\} = 0$$

with coadjoint orbits

$$\mathcal{M}_a = \left\{(M, \Gamma) \in \mathbf{R}^6 \, : \, \langle \Gamma, \Gamma \rangle = 1, \, \langle \Gamma, M \rangle = a \right\} \,,$$

and on each symplectic leaf (1) is Hamiltonian with Hamiltonian function the energy of the body (see [21])

$$E = \tfrac{1}{2} \langle \Omega, M \rangle - \langle \chi, \Gamma \rangle \,.$$

Further we shall be interested in the case when the body is symmetric about an axis through the center of gravity and the fixed point – the so-called Lagrange top [17, p. 253]. This is equivalent to the conditions $I_1 = I_2$ and $\chi = (0, 0, \chi_3)$. Without loss of generality we may also suppose that $\chi_3 / I_1 = 1$, and if we put $m = (I_3 - I_2)/I_1$ then (1) takes the form

(2)
$$\begin{aligned} \dot\Omega_1 &= -m\,\Omega_2\Omega_3 - \Gamma_2 & \dot\Gamma_1 &= \Gamma_2\Omega_3 - \Gamma_3\Omega_2 \\ \dot\Omega_2 &= m\,\Omega_3\Omega_1 + \Gamma_1 & \dot\Gamma_2 &= \Gamma_3\Omega_1 - \Gamma_1\Omega_3 \\ \dot\Omega_3 &= 0 & \dot\Gamma_3 &= \Gamma_1\Omega_2 - \Gamma_2\Omega_1 \end{aligned}$$

with first integrals

$$\begin{aligned} H_1 &= \Gamma_1^2 + \Gamma_2^2 + \Gamma_3^2 \\ H_2 &= \Omega_1\Gamma_1 + \Omega_2\Gamma_2 + (1+m)\Omega_3\Gamma_3 \\ E = H_3 &= \tfrac{1}{2}\left(\Omega_1^2 + \Omega_2^2 + (1+m)\Omega_3^2\right) - \Gamma_3 \,. \end{aligned}$$

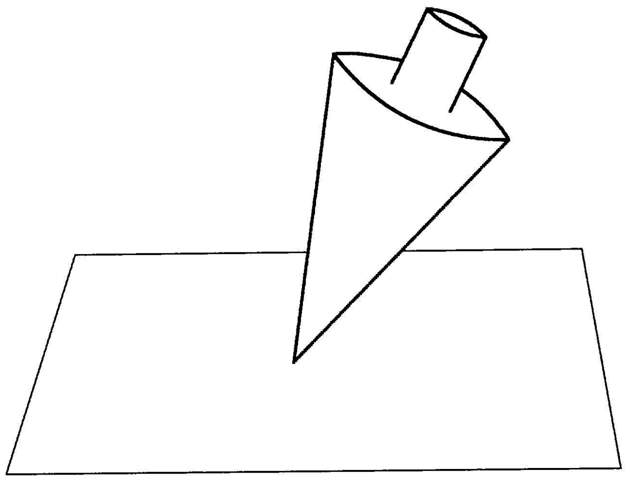

FIGURE 1
The Lagrange top

Due to the symmetry of the body there is an additional integral of motion,

$$H_4 = \Omega_3 \,,$$

which makes (2) Liouville integrable on the symplectic leaf

$$\mathcal{M}_a = \left\{(\Omega,\Gamma) \in \mathbf{R}^6 \,:\, \Gamma_1^2+\Gamma_2^2+\Gamma_3^2 = 1 \,,\ \Omega_1\Gamma_1+\Omega_2\Gamma_2+(1+m)\Omega_3\Gamma_3 = a\right\}.$$

The Hamiltonian vector field generated by H_4 on $\mathcal{M}_a$ is given by

(3)
$$\begin{aligned} \dot\Omega_1 &= \Omega_2 & \dot\Gamma_1 &= \Gamma_2 \\ \dot\Omega_2 &= -\Omega_1 & \dot\Gamma_2 &= -\Gamma_1 \\ \dot\Omega_3 &= 0 & \dot\Gamma_3 &= 0 \end{aligned}$$

and it represents uniform rotations about the symmetry axis through the center of gravity and the fixed point in space.

The Lagrange top is one of the most classical examples of integrable systems and it appears in almost all papers on this subject. The explicit formulae for the position of the body in space ($\Gamma_1, \Gamma_2, \Gamma_3$ in our case) were found by Jacobi [15, p. 503–505]. In the last twenty years most of the integrable problems of classical mechanics were revisited by making use of algebro-geometric techniques. From this point of view the Lagrange top takes a somewhat singular place – the results available are either incomplete, or inexact, or even wrong. Consider the complexified group of rotations $\mathbf{C}^* \sim \mathbf{C}/2\pi i\mathbf{Z}$ defined by the flow of the vector field (3). It acts freely on the generic complex invariant level set

$$T_h = \left\{(\Omega,\Gamma)\in\mathbf{C}^6 \;:\; H_1(\Omega,\Gamma)=1, H_2(\Omega,\Gamma)=h_2, H_3(\Omega,\Gamma)=h_3, H_4(\Omega,\Gamma)=h_4\right\}$$

and it is classically known that the quotient manifold $T_h/\mathbf{C}^*$ is an elliptic curve. The starting point of the present article is the observation that, generically, the algebraic manifold T_h *is not isomorphic* to a direct product of the curve $T_h/\mathbf{C}^*$ and $\mathbf{C}^*$ (although as a topological manifold it is). Let us explain first the algebraic structure of the invariant level set T_h. If $\Lambda \subset \mathbf{C}^2$ is a rank three lattice

$$(4) \qquad \Lambda = \mathbf{Z}\begin{pmatrix} 2\pi i \\ 0 \end{pmatrix} \oplus \mathbf{Z}\begin{pmatrix} 0 \\ 2\pi i \end{pmatrix} \oplus \mathbf{Z}\begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix}, \qquad \mathrm{Re}(\tau_1) < 0$$

then $\mathbf{C}^2/\Lambda$ is a non-compact *algebraic group* and it can be considered as a (non-trivial) extension of the elliptic curve $\mathbf{C}/\{2\pi i\mathbf{Z}\oplus\tau_1\mathbf{Z}\}$ by $\mathbf{C}^* \sim \mathbf{C}/2\pi i\mathbf{Z}$:

$$(5) \qquad 0 \to \mathbf{C}/2\pi i\mathbf{Z} \to \mathbf{C}^2/\Lambda \xrightarrow{\phi} \mathbf{C}/\left\{2\pi i\mathbf{Z} \oplus \tau_1\mathbf{Z}\right\} \to 0, \qquad \phi(z_1, z_2) = z_1 .$$

We prove that, for generic h_i, the complex invariant level set T_h of the Lagrange top is biholomorphic to (an affine part of) $\mathbf{C}^2/\Lambda$. The algebraic group $\mathbf{C}^2/\Lambda$ turns out to be the generalized Jacobian of an elliptic curve with two points identified. This curve, say C, is the spectral curve of a Lax pair for the Lagrange top, found first by Adler and van Moerbeke [1] and its Jacobian $\mathrm{Jac}(C) = \mathbf{C}/\{2\pi i\mathbf{Z} \oplus \tau_1\mathbf{Z}\}$ is a curve found first by ... Lagrange. Further we prove that the flows (2), (3) define translation invariant vector fields on $\mathbf{C}^2/\Lambda$ which means that our system is algebraically completely integrable.

Let us compare the above to the classical Lagrange linearization on an elliptic curve [17] (see also [1, 21, 24, 3, 2]). It is well known that, due to the symmetry of the body, the system (2) is invariant under rotations about the axis of symmetry. These rotations are given by the flow of (3) which commutes with the flow of the Lagrange top. Thus we have a well defined $\mathbf{C}^*$ action on the complex invariant level set $T_h \sim \mathbf{C}^2/\Lambda$ and a well defined (factored) flow on $T_h/\mathbf{C}^*$. Lagrange noted around 1788 that this factorization amounts to eliminating the variables $\Omega_1, \Omega_2, \Gamma_1, \Gamma_2$, so he obtained a single autonomous differential equation for the nutation θ, where $\Gamma_3 = \cos\theta$ [17, p. 254] (nutation is the inclination of the symmetry axis of the body to the vertical). Finally it is seen from this equation that $\Gamma_3(t)$ is, up to an addition and a multiplication by a constant, the Weierstrass elliptic function $\wp(t)$. Thus Lagrange linearized the complex flow of the Lagrange top on an elliptic curve. This curve happens to be the Jacobian $J(C)$ of the spectral curve C of Adler and van Moerbeke and is identified with $\mathbf{C}/\{2\pi i\mathbf{Z} \oplus \tau_1\mathbf{Z}\}$ in (5). The kernel of the map ϕ is just the circle action $\mathbf{C}^* \sim \mathbf{C}/2\pi i\mathbf{Z}$ defined by (3), so the linear vector field (3) is projected under ϕ onto the zero vector field on $\mathrm{Jac}(C) = \mathbf{C}/\{2\pi i\mathbf{Z} \oplus \tau_1\mathbf{Z}\}$.

To summarize in modern language, Lagrange's computation shows that *the generic invariant level set T_h of the Lagrange top is an extension of an elliptic curve $C \sim \mathrm{Jac}(C)$ by $\mathbf{C}^*$ and the flow is projected on this curve into a well defined linear flow*. This is, however, a very vague description of $T_h \sim \mathbf{C}^2/\Lambda$. Indeed, although the fibration

$$\mathbf{C}^2/\Lambda \xrightarrow{\phi} \mathrm{Jac}(C) = \mathbf{C}/\{2\pi i\mathbf{Z} \oplus \tau_1\mathbf{Z}\} \tag{6}$$

is topologically trivial, it is not algebraically trivial, and to know its *type* we need the parameter τ_2 defined in (4) (cf. [23]). As the general solution of (2) lives on $\mathbf{C}^2/\Lambda$ then, contrary to what is often asserted, it cannot be expressed in terms of elliptic functions and exponentials. It is even less true that "the flow of the Lagrange top lives on a complex 2-dimensional cylinder with generator the line $z = 0$" as claimed in [21, p. 232].

The algebraic description of the Lagrange top is carried out in Section 2 (Theorem 2.2). The Lax pair is used first in Section 3 where we construct the corresponding Baker-Akhiezer function. This implies explicit formulae for the general solution of the Lagrange top which complete and simplify the classical formulae due to Jacobi [15, p. 503–505] for $\Gamma_1, \Gamma_2, \Gamma_3$ and Klein and Sommerfeld [16, p. 436] for the angular velocities (Theorem 3.6).

In Section 4 we study reality conditions on the (complex) solutions. Besides the usual real structure of the Lagrange top given by complex conjugation there is a second natural real structure induced by the eigenvalue map of the corresponding Lax pair representation. It turns out that these two structures coincide on $\mathrm{Jac}(C)$ but are different on $\mathbf{C}^2/\Lambda$ (and hence on T_h). The corresponding real level sets are described in Theorem 4.2. This makes clear the relation between the real structure of the curve C, its Jacobian $\mathrm{Jac}(C)$ and the real level set $T_h^{\mathbf{R}}$ (a question raised in [2] and [3, p. 37]).

The results obtained in the present paper lead to the following unexpected observation: the real solutions of the Lagrange top corresponding to its two real structures provide one-gap solutions of the nonlinear Schrödinger equation (Proposition 5.1)

$$(NLS^{\pm}) \qquad u_{xx} = iu_t \pm 2|u|^2 u\,.$$

Finally, for the convenience of the reader, we give in the Appendix a brief account of some more or less well known results concerning the linearization of the Lagrange top on an elliptic curve.

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2. ALGEBRAIC STRUCTURE

Let $\check{C}$ be the affine curve $\{\mu^2 = f(\lambda)\}$ where f is a degree 4 polynomial without double roots. We denote by C the completed and normalized curve $\check{C}$. Thus C is a compact Riemann surface, such that $C = \check{C} \cup \infty^+ \cup \infty^-$, where $\infty^{\pm}$ are two distinct "infinite" points on C. Consider the effective divisor $m = \infty^+ + \infty^-$ on C and let $J_m(C)$ be the generalized Jacobian of the elliptic curve C relative to m. Following [23] we shall call m a modulus. We shall denote also $J(C;\infty^{\pm}) = J_m(C)$. Recall that the usual Jacobian

$$J(C) = \mathrm{Div}^0(C) \big/ \sim$$

is the additive group $\mathrm{Div}^0(C)$ of degree zero divisors on C modulo the equivalence relation $\sim$. We have $D_1 \sim D_2$ if and only if there exists a meromorphic function f on C such that $(f) = D_1 - D_2$. Similarly the generalized Jacobian

$$J(C;\infty^{\pm}) = \mathrm{Div}^0(\check{C}) \big/ \overset{m}{\sim}$$

is the additive group $\mathrm{Div}^0(\check{C})$ of degree zero divisors on $\check{C}$ modulo the equivalence relation $\overset{m}{\sim}$. We have $D_1 \overset{m}{\sim} D_2$ if and only if there exists a meromorphic function f on C such that $f(\infty^+) = f(\infty^-) = 1$ and $(f) = D_1 - D_2$. The generalized Jacobian $J(C;\infty^{\pm})$ is thus obtained as a $\mathbf{C}^*$-extension of the usual Jacobian $J(C)$ (isomorphic to C). This means that there is an exact sequence of groups

$$(7) \qquad 0 \xrightarrow{\exp} \mathbf{C}^* \xrightarrow{v} J(C;\infty^{\pm}) \xrightarrow{\phi} J(C) \to 0\,.$$