# 2. Algebraic structure

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real structures provide one-gap solutions of the nonlinear Schrödinger equation (Proposition 5.1)

$$(NLS^{\pm}) \qquad u_{xx} = iu_t \pm 2|u|^2 u\,.$$

Finally, for the convenience of the reader, we give in the Appendix a brief account of some more or less well known results concerning the linearization of the Lagrange top on an elliptic curve.

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## 2. ALGEBRAIC STRUCTURE

Let $\check{C}$ be the affine curve $\{\mu^2 = f(\lambda)\}$ where $f$ is a degree 4 polynomial without double roots. We denote by $C$ the completed and normalized curve $\check{C}$. Thus $C$ is a compact Riemann surface, such that $C = \check{C} \cup \infty^+ \cup \infty^-$, where $\infty^{\pm}$ are two distinct "infinite" points on $C$. Consider the effective divisor $m = \infty^+ + \infty^-$ on $C$ and let $J_m(C)$ be the generalized Jacobian of the elliptic curve $C$ relative to $m$. Following [23] we shall call $m$ a modulus. We shall denote also $J(C;\infty^{\pm}) = J_m(C)$. Recall that the usual Jacobian

$$J(C) = \mathrm{Div}^0(C)\,/\sim$$

is the additive group $\mathrm{Div}^0(C)$ of degree zero divisors on $C$ modulo the equivalence relation $\sim$ . We have $D_1 \sim D_2$ if and only if there exists a meromorphic function $f$ on $C$ such that $(f) = D_1 - D_2$. Similarly the generalized Jacobian

$$J(C;\infty^{\pm}) = \mathrm{Div}^0(\check{C})\,/\overset{m}{\sim}$$

is the additive group $\mathrm{Div}^0(\check{C})$ of degree zero divisors on $\check{C}$ modulo the equivalence relation $\overset{m}{\sim}$. We have $D_1 \overset{m}{\sim} D_2$ if and only if there exists a meromorphic function $f$ on $C$ such that $f(\infty^+) = f(\infty^-) = 1$ and $(f) = D_1 - D_2$. The generalized Jacobian $J(C;\infty^{\pm})$ is thus obtained as a $\mathbf{C}^*$-extension of the usual Jacobian $J(C)$ (isomorphic to $C$). This means that there is an exact sequence of groups

$$0 \xrightarrow{\exp} \mathbf{C}^* \xrightarrow{v} J(C;\infty^{\pm}) \xrightarrow{\phi} J(C) \to 0\,. \tag{7}$$

The map $\phi$ is induced by the inclusion $\check{C} \subset C$ and $v(r) \in J(C;\infty^{\pm})$, $r \neq 0$, is the divisor of any meromorphic function $f$ on $C$ satisfying $f(\infty^{+})/f(\infty^{-}) = r$ [10, p. 55].

As an analytic manifold $J(C;\infty^{\pm})$ is

$$\mathbf{C}^2/\Lambda \sim H^0\big(C,\Omega^1(\infty^{+}+\infty^{-})\big)^* \,/H_1(\check{C},\mathbf{Z}) \tag{8}$$

where the lattice $\Lambda$ is generated by the three vectors

$$\Lambda_1 = \begin{pmatrix} \int_{A_1} \frac{d\lambda}{\mu} \\ \int_{A_1} \frac{\lambda\, d\lambda}{\mu} \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} \int_{A_2} \frac{d\lambda}{\mu} \\ \int_{A_2} \frac{\lambda\, d\lambda}{\mu} \end{pmatrix}, \quad \Lambda_3 = \begin{pmatrix} \int_{B_1} \frac{d\lambda}{\mu} \\ \int_{B_1} \frac{\lambda\, d\lambda}{\mu} \end{pmatrix} \tag{9}$$

and the cycles $A_1, A_2, B_1$ form a basis of the first homology group $H_1(\check{C},\mathbf{Z})$ as in Figure 2. It is seen that the period lattice $\Lambda$ may be obtained by pinching a non-zero homology cycle of a genus two Riemann surface to a point $\infty^{\pm}$ (Figure 2). This is expressed by saying that $J(C;\infty^{\pm})$ is the Jacobian of the elliptic curve $C$ with two points $\infty^{+}$ and $\infty^{-}$ identified [10].

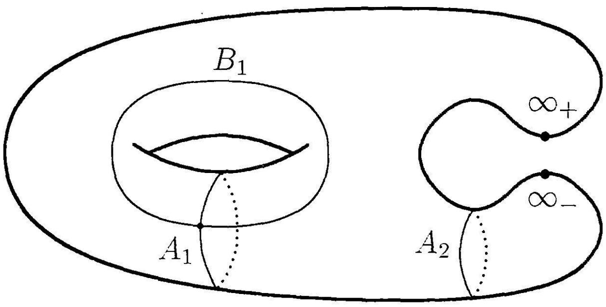


FIGURE 2
The canonical homology basis of the affine curve $C$

For further use, note also that

$$\phi\colon J(C;\infty^{\pm}) \to J(C), \quad \phi\colon \mathbf{C}^2/\Lambda \to \mathbf{C}/\phi(\Lambda) \tag{10}$$

is just the first projection $\phi(z_1, z_2) = z_1$. As

$$\phi(\Lambda_2) = \int_{A_2} \frac{d\lambda}{\mu} = 0,$$

$\phi(\Lambda)$ is generated by $\phi(\Lambda_1)$ and $\phi(\Lambda_3)$, and

$$\operatorname{Ker} \phi = \mathbf{C}\Big/\Big\{\mathbf{Z}\int_{A_2} \frac{\lambda\, d\lambda}{\mu}\Big\} \sim \mathbf{C}^*.$$

As an analytic manifold the usual Jacobian $J(C)$ is

$$\mathbf{C}/\phi(\Lambda) \sim H^0(C,\Omega^1)^* \,/H_1(C,\mathbf{Z}).$$

In contrast to the usual Jacobian $J(C)$, the generalized Jacobian $\mathbf{C}^2/\Lambda$ is a *non-compact* algebraic group. For any $p \in J(C)$ define also the divisor $D_p = \phi^{-1}(p) \subset J(C; \infty^\pm)$.

An explicit embedding of a Zariski open subset of $J(C; \infty^\pm)$ in $\mathbf{C}^6$ is constructed by the following classical construction due to Jacobi (see Mumford [18]). Let

$$f(\lambda) = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 \tag{11}$$

be a polynomial without double roots and define the polynomials

$$U(\lambda) = \lambda^2 + u_1\lambda + u_2, \qquad V(\lambda) = v_1\lambda + v_2, \qquad W(\lambda) = \lambda^2 + w_1\lambda + w_2. \tag{12}$$

Let $T_C$ be the set of Jacobi polynomials (12) satisfying the relation

$$f(\lambda) - V^2(\lambda) = U(\lambda)W(\lambda). \tag{13}$$

More explicitly, let us expand

$$f - V^2 - UW = \sum_{i=0}^{3} b_i(u_1, u_2, v_1, v_2, w_1, w_2)\,\lambda^i,$$

$$b_3 = a_1 - u_1 - w_1, \qquad b_2 = a_2 - u_2 - w_2 - u_1w_1 - v_1^2,$$

$$b_1 = a_3 - u_1w_2 - u_2w_1 - 2v_1v_2, \qquad b_0 = a_4 - u_2w_2 - v_2^2.$$

If we take $u_i, v_j, w_k$ as coordinates in $\mathbf{C}^6$ then $T_C$ is just the zero locus $V(b_0, b_1, b_2, b_3)$ as a subset of $\mathbf{C}^6$

$$T_C = \big\{(u, v, w) \in \mathbf{C}^6 : u_1 + w_1 = a_1, \quad u_2 + w_2 + u_1w_1 + v_1^2 = a_2, \quad u_1w_2 + u_2w_1 + 2v_1v_2 = a_3, \quad u_2w_2 + v_2^2 = a_4\big\}.$$

PROPOSITION 2.1. *If $f(\lambda)$ is a polynomial without double roots, then*

(i) *$T_C$ is a smooth affine variety isomorphic to $J(C; \infty^\pm) \setminus D_p$ for some $p \in J(C)$;*

(ii) *any translation invariant vector field on the generalized Jacobian $J(C; \infty^\pm)$ of the curve $C$ can be written (up to multiplication by a non-zero constant) in the following Lax pair form*

$$2\sqrt{-1}\,\frac{d}{dt}A(\lambda) = \left[A(\lambda), \frac{A(a)}{\lambda - a}\right] \tag{14}$$

*where*

$$A(\lambda) = \begin{pmatrix} V(\lambda) & U(\lambda) \\ W(\lambda) & -V(\lambda) \end{pmatrix}, \tag{15}$$

*$a \in \mathbf{C}$, and $U, V, W$ are the Jacobi polynomials* (12).

*Equivalently, if $D = P_1 + P_2 \in \mathrm{Div}^2(\check{C})$, where $P_i = (\lambda_i, \mu_i)$, $i = 1, 2$, then* (14) *can be written as*

$$
\begin{aligned}
&\frac{d\lambda_1}{\sqrt{f(\lambda_1)}} + \frac{d\lambda_2}{\sqrt{f(\lambda_2)}} = -\sqrt{-1}\, dt \\
&\frac{\lambda_1 d\lambda_1}{\sqrt{f(\lambda_1)}} + \frac{\lambda_2 d\lambda_2}{\sqrt{f(\lambda_2)}} = -a\sqrt{-1}\, dt\,.
\end{aligned}
\tag{16}
$$

REMARK. Note that $a = \infty$ also makes sense. The corresponding vector field is obtained by changing the time as $t \to t/a$ and letting $a \to \infty$. Thus (14) becomes

$$
2\sqrt{-1}\,\frac{d}{dt} A(\lambda) = \left[A(\lambda)\,,\, A_\infty\right], \qquad A_\infty = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}
\tag{17}
$$

and (16) becomes

$$
\begin{aligned}
&\frac{d\lambda_1}{\sqrt{f(\lambda_1)}} + \frac{d\lambda_1}{\sqrt{f(\lambda_1)}} = 0 \\
&\frac{\lambda_1 d\lambda_1}{\sqrt{f(\lambda_1)}} + \frac{\lambda_2 d\lambda_2}{\sqrt{f(\lambda_2)}} = -\sqrt{-1}\, dt\,.
\end{aligned}
\tag{18}
$$

The proof of part (i) of the above proposition can be found in Previato [20] (see also Mumford [18]). It is also proved there that a translation invariant vector field $\frac{d}{d\epsilon}$ on the generalized Jacobian $J(C; \infty^\pm)$ which is induced by the tangent vector

$$
\frac{d}{d\epsilon} \lambda\Big|_{\lambda = a} = \sqrt{f(a)}
\tag{19}
$$

on $C$ via the Abel map $C \to J(C; \infty^\pm)$, can be written as

$$
\frac{d}{d\epsilon} U(\lambda) = \frac{V(a)U(\lambda) - U(a)V(\lambda)}{\lambda - a}
\tag{20}
$$

$$
\frac{d}{d\epsilon} W(\lambda) = -\frac{V(a)W(\lambda) - W(a)V(\lambda)}{\lambda - a}
\tag{21}
$$

$$
\frac{d}{d\epsilon} V(\lambda) = \frac{U(a)W(\lambda) - W(a)U(\lambda)}{2(\lambda - a)}\,.
\tag{22}
$$

Our final remark is that the translation invariant vector fields (20), (21) and (22), which we denote further by $\frac{d}{dt}$, can be written in the following Lax pair form (suggested by Beauville [6, Example 1.5]):

$$-2\frac{d}{dt}A(\lambda) = \left[A(\lambda), \frac{A(a)}{\lambda - a}\right],$$

where

$$A(\lambda) = \begin{pmatrix} V(\lambda) & U(\lambda) \\ W(\lambda) & -V(\lambda) \end{pmatrix}.$$

By (19) the direction of the constant tangent vector computed above is

$$\left(\frac{\dot\lambda}{\sqrt{f(a)}}, \frac{a\dot\lambda}{\sqrt{f(a)}}\right) = (1, a),$$

which proves (16). This completes the proof of Proposition 2.1. □

Next we apply Proposition 2.1 to the Lagrange top (2). Let $C_h$ be the curve $C$ as above, where

$$\begin{aligned} a_1 &= 2(1+m)h_4, & a_2 &= 2h_3 + m(m+1)h_4^2, \\ a_3 &= -2h_2, & a_4 &= h_1 = 1, \end{aligned} \tag{23}$$

so

$$\check{C}_h = \left\{\mu^2 = \lambda^4 + 2(1+m)h_4\lambda^3 + \left(2h_3 + m(m+1)h_4^2\right)\lambda^2 - 2h_2\lambda + 1\right\}. \tag{24}$$

Consider the complex invariant level set of the Lagrange top (2)

$$T_h = \left\{(\Omega,\Gamma)\in \mathbf{C}^6 : H_1(\Omega,\Gamma)=1,\ H_2(\Omega,\Gamma)=h_2,\ H_3(\Omega,\Gamma)=h_3,\ H_4(\Omega,\Gamma)=h_4\right\}$$

and the associated "bifurcation set"

$$\mathbf{B} = \left\{h \in \mathbf{C}^3 : \text{discriminant}\left(f(\lambda)\right) = 0\right\}.$$

It is a straightforward computation to check that the linear change of variables

$$\begin{aligned} u_1 &= (1+m)\Omega_3 - i\Omega_2 & u_2 &= -\Gamma_3 + i\Gamma_2 \\ w_1 &= (1+m)\Omega_3 + i\Omega_2 & w_2 &= -\Gamma_3 - i\Gamma_2 \\ v_1 &= \Omega_1 & v_2 &= -\Gamma_1 \end{aligned} \tag{25}$$

(with $i = \sqrt{-1}$) identifies $T_C$ and $T_h$. Further, as

$$\left[A(\lambda), \frac{A(a)}{\lambda - a}\right] = \left[A(\lambda), \frac{A(a) - A(\lambda)}{\lambda - a}\right] = \left[A(\lambda), \begin{pmatrix} -v_1 & -a - u_1 - \lambda \\ -a - w_1 - \lambda & v_1 \end{pmatrix}\right],$$

the vector field (2) is obtained by substituting $a = -m\Omega_3$ in (14) and using the change of variables (25) (note that $\Omega_3$ is a constant of motion). Similarly the vector field (3) is obtained by substituting $a = \infty$ (see the remark after Proposition 2.1).

To sum up, we have proved the following

THEOREM 2.2. *If* $h \notin \mathbf{B}$, *then*

(i) *the complex invariant level set* $T_h$ *of the Lagrange top is a smooth complex manifold biholomorphic to* $J(C_h;\infty^{\pm})\backslash D_\infty$ *where* $D_\infty = \phi^{-1}(p)$ *for some* $p \in J(C_h)$ *and* $J(C_h;\infty^{\pm})$ *is the generalized Jacobian of the elliptic curve* $C_h$ *with two points at "infinity" identified;*

(ii) *the Hamiltonian flows of the Lagrange top* (2), (3) *restricted to* $T_h$ *induce linear flows on* $J(C_h;\infty^{\pm})$. *The corresponding vector fields* (2) *and* (3) *have a Lax pair representation obtained from the Lax pair* (14) *by substituting* $a = -m\Omega_3$ *and* $a = \infty$ *respectively, and using the change of variables* (25).

According to the above theorem the Lagrange top is an algebraically completely integrable system in the sense of Mumford [18, p. 353]. Clearly any linear flow on $J(C_h;\infty^{\pm})$ maps under $\phi$ (7) into a linear flow on the usual Jacobian $J(C_h)$. This is expressed by the fact that the variable $\Gamma_3$ which describes the nutation of the body is an elliptic function in time. It was known to Lagrange [17] who deduced the differential equation satisfied by $\Gamma_3$. The real version of Theorem 2.2 will be explained in Section 4.

To the end of this section we compare the Lax pair (14) and the Lax pair for the Lagrange top obtained earlier by Adler and van Moerbeke [1]. Namely, if we identify the Lie algebras $(\mathbf{R}^3, \wedge)$ and $(\mathfrak{so}(3), [\,.\,,\,.\,])$ by the Lie algebra isomorphism

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix},$$

then (2) can be written in the following equivalent Lax pair form [1]

$$\frac{d}{dt}\left(\lambda^2\chi + \lambda M - \Gamma\right) = \left[\lambda^2\chi + \lambda M - \Gamma\,,\, \lambda\chi + \Omega\right], \tag{26}$$

where

$$\Omega = (\Omega_1, \Omega_2, \Omega_3),\ M = \left(\Omega_1, \Omega_2, (1+m)\Omega_3\right),\ \Gamma = (\Gamma_1, \Gamma_2, \Gamma_3),\ \chi = (0, 0, 1).$$

The Lax pair representation of (3) is given by

$$\frac{d}{dt}\left(\lambda^2\chi + \lambda M - \Gamma\right) = \left[\lambda^2\chi + \lambda M - \Gamma\,,\, \chi\right]. \tag{27}$$

Both Lax pairs (26), (27) can be also written in the Beauville form

$$-\frac{d}{dt}A(\lambda) = \left[A(\lambda), \frac{A(a)}{\lambda - a}\right] \tag{28}$$

where $A(\lambda) = \lambda^2\chi + \lambda M - \Gamma$. Indeed,

$$\left[A(\lambda), \frac{A(a)}{\lambda - a}\right] = \left[A(\lambda), \frac{A(a) - A(\lambda)}{\lambda - a}\right] = -\left[A(\lambda), \lambda\chi + a\chi + M\right].$$

Now (26) is obtained by replacing as before $a = -m\Omega_3$, and (27) is obtained by letting $a \to \infty$.

Clearly the Lax pair (14) from Proposition 2.1 and (26), (28) are equivalent in the sense that they define one and the same vector field. We can identify them over $\mathbf{C}$ by making use of the isomorphism of the Lie algebras $\mathfrak{so}(3, \mathbf{C})$ and $\mathfrak{sl}(2, \mathbf{C})$ given by

$$\begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} \to \frac{1}{\sqrt{2}} \begin{pmatrix} \epsilon x & \epsilon z + \bar{\epsilon} y \\ \epsilon z - \bar{\epsilon} y & -\epsilon x \end{pmatrix}, \qquad \epsilon = \exp \frac{\sqrt{-1}\pi}{4}.$$

Note, however, the following difference. The spectral curve of (26) is reducible

$$\det\left(\lambda^2\chi + \lambda M + \Gamma - \mu I\right) = -\mu\left(\mu^2 + f(\lambda)\right) = 0,$$

$$f(\lambda) = \lambda^4 + 2(1+m)h_4\lambda^3 + \left(2h_3 + m(m+1)h_4^2\right)\lambda^2 - 2h_2\lambda + 1,$$

but the spectral curve of (14) is not

$$\det\left(A(\lambda) - \mu I\right) = \mu^2 - V^2 - UW = \mu^2 - f(\lambda) = 0.$$

The last observation will be of some importance for the next section. Earlier Adler and van Moerbeke [1, p. 351] proposed to linearize the Lagrange top on an elliptic curve by introducing first a small parameter $\epsilon$ in the corresponding $\mathfrak{so}(3)$ Lax pair. The new system has the advantage of having an irreducible genus 4 spectral curve $C_\epsilon$ which fits the general theory, so we can just "take the limit" $\epsilon \to 0$. This computation, reproduced in [21] and used in [22], is however erroneous.

By abuse of notation we call the curve $\widetilde{C}_h = \{\mu^2 + f(\lambda) = 0\}$ with an antiholomorphic involution $(\lambda, \mu) \to (\bar{\lambda}, \bar{\mu})$, the *spectral curve* of the Lax pair (26). The curve $\widetilde{C}_h$ is real isomorphic to the curve $C_h = \left\{\mu^2 = f(\lambda)\right\}$, equipped with an antiholomorphic involution $(\lambda, \mu) \to (\bar{\lambda}, -\bar{\mu})$, so without loss of generality we shall write $\widetilde{C}_h = C_h$.