AN ASYMPTOTIC FREIHEITSSATZ FOR FINITELY GENERATED GROUPS

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AN ASYMPTOTIC FREIHEITSSATZ FOR FINITELY GENERATED GROUPS

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ABSTRACT. Given two fixed integers $k \geq 2$ and $l \geq 3$, let $\Gamma = \langle X \mid R \rangle$ be a presentation of the group Γ with $k = \#X$ generators and $l = \#R$ relations. We show that the following property of presentations of groups is generic in the sense of Gromov: for any $y \in X$, the subgroup of Γ generated by $X - \{y\}$ is free of rank $k-1$. This gives some generic estimates for the spectral radius of the adjacency operator in the Cayley graph of Γ relative to the generating system $S = X \cup X^{-1}$.

1. INTRODUCTION

The existence of free subgroups in some finitely generated group Γ gives some information about the structure of Γ. For example, it implies that Γ is non-amenable, and in particular that Γ has exponential growth. There are several results which ensure that various groups do have non-abelian free subgroups. For example:

THEOREM (Tits's alternative [15]). *Let Γ be a finitely generated linear group. Then either Γ is almost solvable or Γ contains a free subgroup on two generators.*

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THEOREM (Magnus's Freiheitssatz [12]). *Let* $\Gamma = \langle X \,|\, r \rangle$ *be a one relator group,* $x_0 \in X$ *be a generator of* Γ *that appears in the relation* r *and* r *be a cyclically reduced word in the free group* $\mathbf{F}_X$ *generated by* X*; then* $X - \{x_0\}$ *freely generates a free group in* Γ.

Our purpose in this work is to measure in some sense how frequent it is for a presentation $\Gamma = \langle X \,|\, R \rangle$ to be such that a proper subset of X is free in Γ. We prove the following result:

THEOREM 1.1. *Let* $\Gamma = \langle X \,|\, R \rangle$ *be a finite presentation with* k *generators,* l *relations and any fixed* x_0 *in* X. *Then the fact that* $X - \{x_0\}$ *freely generates a free group in* Γ *is generic in the sense of Gromov.*

The key idea is contained in proposition 4.1. Roughly speaking, if you choose at random l long relations and if the presentation satisfies a Dehn algorithm, then every generator x_0 will appear in every sufficiently long subword of every relation and hence it will appear in every product of conjugates of relations. So $X - \{x_0\}$ generates a free group in Γ.

In [6], the first author has shown that "X generates a free semi-group" is generic and that this implies bounds on the spectrum of the adjacency operator associated to the oriented Cayley graph of Γ relative to X. In section 5 below, we consider the adjacency operator h_S of the Cayley graph of Γ relative to $S = X \cup X^{-1}$, and we prove similarly estimates on the norm of h_S.

After finishing this paper we discovered that a result similar to Theorem 1.1 has been proved, using different methods, by G. Arzhantseva and A. Ol′shanskii in [1]. They employed a slightly different definition of the genericity and they proved that the small cancellation condition $C'(\lambda)$ is generic with respect to this new definition.

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2. SOME DEFINITIONS

First, we recall what Gromov's genericity is.

DEFINITION (Champetier). Consider two integers $k \geq 2$, $l \geq 1$, a set X of k generators and a property P of group presentations with X as generating system and with l relations. For integers $n_1, \ldots, n_l \geq 1$, let $Pr(X, n_1, \ldots, n_l)$ denote the finite set of presentations $\langle X \mid r_1, \ldots, r_l \rangle$ where r_i is a cyclically reduced relation in the generators of X which is of length $|r_i| = n_i$ $(1 \leq i \leq l)$.

Then P is said to be generic in the sense of Gromov if the ratio

$$\frac{\#\{\langle X \mid R\rangle \in Pr(X, n_1, \ldots, n_l) \mid \langle X \mid R\rangle \text{ satisfies } P\}}{\#Pr(X, n_1, \ldots, n_l)}$$

tends to 1 when $\min_{i=1,\ldots,l} n_i \to +\infty$.

For example, being a hyperbolic group is a generic property. This was proved independently by Champetier [5] and Ol′shanskii [13].

One tool we need is small cancellation theory. Let $\langle X \mid R\rangle$ be a presentation of a group Γ. Denote by R^* the set of cyclic conjugates of elements of R and of their inverses.

DEFINITION 2.1. Let $\Gamma = \langle X \mid R\rangle$ be a finitely presented group. A *piece* is a prefix u common to at least two distincts elements in R^* (by prefix, we mean every non empty initial part of a word; in particular a word is a particular prefix for itself).

Fix $\lambda \in]0,1[$. The presentation $\langle X \mid R\rangle$ satisfies the *small cancellation condition* $C'(\lambda)$ if the following inequality holds: $|u| < \lambda |r|$ for every $r \in R^*$ and for every prefix u of r which is a piece.

DEFINITION 2.2. A group $\Gamma = \langle X \mid R\rangle$ satisfies a *Dehn algorithm* if, for every non trivial reduced word $\omega \in \mathbf{F}_X$ representing 1 in Γ, there exists a prefix u of some word $r \in R^*$ such that u is a subword of ω and $|u| > \frac{1}{2}|r|$.

It is known that groups satisfying the small cancellation condition $C'(1/6)$ also admit a Dehn algorithm (see Theorem 4.4, Chapter V in [11] or Theorem 25 in [14]). On the other hand Gromov proves that groups with a Dehn algorithm are hyperbolic (see [8, Theorem 2.3.D]).

In Proposition 4.1 below, $C'(1/6)$ is one of the conditions which imply that, for some fixed $x_0 \in X$, $X - \{x_0\}$ generates a free subgroup in Γ.

Let $\langle X \,|\, R \rangle$ be a presentation with k generators and l relations $r_1, \ldots, r_l$. G. Arzhantseva and A. Ol′shanskii proved, in [1], that for any fixed $\lambda > 0$,

$$\lim_{d \to +\infty} \frac{\#\{\langle X \,|\, R \rangle \text{ with } C'(\lambda) \mid \sum_{i=1}^{l} |r_i| = d,\ \ r_i \text{ cyclically reduced}\}}{\#\{\langle X \,|\, R \rangle \mid \sum_{i=1}^{l} |r_i| = d,\ \ r_i \text{ cyclically reduced}\}} = 1 .$$

Unfortunately, even with this result, it is not known if the small cancellation hypothesis is generic, so we need another hypothesis which is generic. Let us recall the definition of Van Kampen diagrams.

DEFINITION 2.3. Let $\omega \in \mathbf{F}_X$ represent the identity in $\Gamma = \langle X \,|\, R \rangle$. Then Δ is a *Van Kampen diagram* of ω if Δ is a planar 2-complex for which the 1-skeleton is a graph, each edge of it being labelled by a element of X or X^{-1} such that when we read the labelling of every 2-cell of the complex, we get a word in R^*, and such that the labelling of the border of the complex Δ is the word ω.

For more details about Van Kampen diagrams, see [14], [3] or [11]. We denote by $I(\Delta)$ (resp. $E(\Delta)$ and $\#(\Delta)$) the number of internal edges of Δ (resp. the number of external edges of Δ and the total number of edges of Δ).

DEFINITION 2.4. The *combinatorial area* of a Van Kampen diagram Δ is the number of its 2-cells. We say that Δ is a reduced diagram of ω if it has the minimal combinatorial area among all diagrams representing ω.

For every $\omega \in \mathbf{F}_X$ representing the identity in $\Gamma = \langle X \,|\, R \rangle$, the existence of such a reduced diagram of ω is proved in [3].

DEFINITION 2.5. For $0 < \theta < 1$, a finite presentation $\langle X \mid R \rangle$ is said to satisfy the *θ-condition*, if for every reduced diagram Δ associated with a reduced word ω in $\mathbf{F}_X$ representing the identity in $\langle X \mid R \rangle$, we have $I(\Delta) < \theta \#(\Delta)$.

In [13], Ol′shanskii showed that for every fixed $\theta > 0$, the property of satisfying a θ-condition is generic.

To prove that result, he needed to introduce the following definition.

DEFINITION 2.6. A reduced diagram is *simple* if every edge is contained in the boundary of a 2-cell of the diagram.

It is clear that every reduced diagram of ω is a disjoint union of simple ones linked by bridges, where a bridge is a finite path of edges which are not in the boundary of a 2-cell, and, because the word ω in $\mathbf{F}_X$ is reduced, each bridge links two simple diagrams. In figure 1 the diagram contains three simple diagrams (D1, D2, D3) and two bridges (B1, B2).

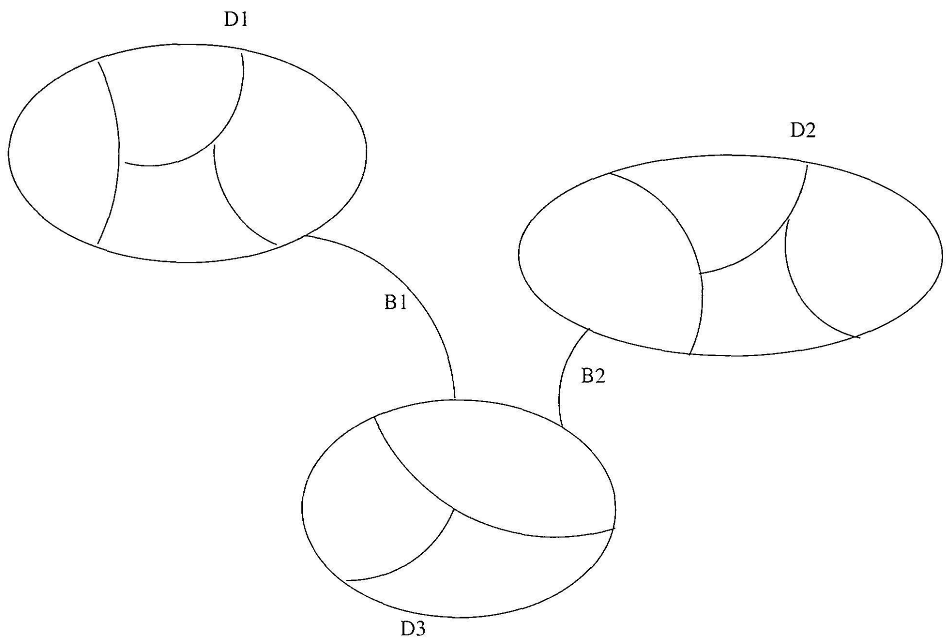

FIGURE 1
A non simple diagram

Let X be a set of generators and $y \in X$. For every reduced word $r \in \mathbf{F}_X$, we denote by $n_y(r)$ the number of occurences of y and y^{-1} in r. For example $n_y(yx^3y^{-2}xy^3) = 6$.

DEFINITION 2.7. Let $\mathbf{F}_X$ be the free group on X with $\#X = k$. For a fixed ϵ with $0 < \epsilon < 1/k$ and $y \in X$, a non trivial reduced word $r \in \mathbf{F}_X$ is (ϵ, y)-balanced if

$$\frac{n_y(r)}{|r|} \geq \epsilon .$$

A presentation $\Gamma = \langle X \mid R \rangle$ is (ϵ, y)-balanced, if every $r \in R$ is *(ϵ, y)-balanced.*

3. About genericity

LEMMA 3.1. *Let* $X = \{x_1, \ldots, x_k, y\}$. *For every* $0 < \epsilon < 1/(k+1)$, *the ratio*

$$\frac{\#\{r \in \mathbf{F}_X \mid |r| = n, r \text{ is } (\epsilon, y)\text{-balanced}\}}{\#\{r \in \mathbf{F}_X \mid |r| = n\}}$$

tends to 1 *when* n *tends* ∞.

Proof. First we want to rephrase the Lemma in terms of generating functions. Let K be any fixed subset of $\mathbf{F}_X$ and $F_K(z,u)$ be the generating function defined by

$$F_K(z,u) = \sum_{r \in K} z^{|r|} u^{n_y(r)} .$$

$F_K(z,u)$ strongly depends on the choice of the generator y. However, as y is fixed throughout the proof and to lighten the notation, we write $F_K(z,u)$ instead of $F_{y,K}(z,u)$.

Defining $c_{n,l}$ and $p_n(l)$ by

$$F_{\mathbf{F}_X}(z,u) = \sum_{r \in \mathbf{F}_X} z^{|r|} u^{n_y(r)} = \sum_{n,l} c_{n,l} z^n u^l \quad \text{and} \quad p_n(l) = \frac{c_{n,l}}{\sum_m c_{n,m}} ,$$

we have to prove that for every $0 < \epsilon < 1/(k+1)$,

$$\lim_{n \to \infty} \sum_{0 \leq l < \epsilon n} p_n(l) = 0 .$$

We want to find an analytical form for $F_{\mathbf{F}_X}(z,u)$.

It is clear that if K_1 and K_2 are disjoint subsets of $\mathbf{F}_X$ then $F_{K_1 \cup K_2}(z,u) = F_{K_1}(z,u) + F_{K_2}(z,u)$.

Let K_1, K_2 be two subsets of F_X; assume that the map $K_1 \times K_2 \to K_1 K_2$ defined by $(\omega_1, \omega_2) \mapsto \omega_1\omega_2$ is one to one and satisfies $|\omega_1\omega_2| = |\omega_1| + |\omega_2|$ for $\omega_i \in K_i$ (where $K_1K_2 = \{\omega_1\omega_2 | \omega_i \in K_i\}$); it is also clear that $F_{K_1K_2}(z,u) = F_{K_1}(z,u)F_{K_2}(z,u)$. This can be extended to a finite product of such K_i's.

First we compute the generating functions of some subsets K of $\mathbf{F}_X$.

- $F_{\{e\}}(z,u) = 1$.
- Denote by $X' = X - \{y\}$. As there are exactly $2k(2k-1)^{n-1}$ reduced words of length $n \geq 1$ in $\mathbf{F}_{X'}$, we obtain $F_{[\mathbf{F}_{X'} - \{e\}]}(z,u) = \frac{2kz}{1-(2k-1)z}$. Set $f(z,u) = F_{[\mathbf{F}_{X'} - \{e\}]}(z,u)$.
- For $\langle y \rangle = \{y^i \mid i \in \mathbf{Z} - \{0\}\}$, we have $F_{\langle y \rangle}(z,u) = \frac{2uz}{1-uz}$, because there are exactly 2 elements $y^{\pm i}$ in $\langle y \rangle$ such that $n_y(y^{\pm i}) = |y^{\pm i}| = i$. Set $h(z,u) = F_{\langle y \rangle}(z,u)$.

Now we can partition $\mathbf{F}_X$ as follows:

$$\mathbf{F}_X = \{e\} \amalg [\mathbf{F}_{X'} - \{e\}] \coprod_{n\geq 0} I_n$$

where

$$I_n = \{\omega_0 y^{i_1} \omega_1 \dots y^{i_{n-1}} \omega_{n-1} y^{i_n} \omega_n \mid \omega_j \in \mathbf{F}_{X'},\ \omega_j \neq e \text{ for } j \neq 0 \text{ or } n, \text{ and } i_j \neq 0\}.$$

It is easy to check that $F_{I_n}(z,u) = (f(z,u)+1)^2\, h(z,u)\, (h(z,u)f(z,u))^{n-1}$. So we obtain that

$$\begin{aligned} F_{\mathbf{F}_X}(z,u) &= 1 + f(z,u) + \sum_{n\geq 1} (f(z,u)+1)^2\, h(z,u)\, (h(z,u)f(z,u))^{n-1} \\ &= (1+f(z,u))\Big(1 + \frac{h(z,u)(f(z,u)+1)}{1-h(z,u)f(z,u)}\Big) \\ &= \frac{(1+z)(1+uz)}{1-(2k-1)z-uz(1+(2k+1)z)}. \end{aligned}$$

Borrowing notation from [2], let $g(z,u) = (1+z)(1+zu)$ and $P(z,u) = 1-(2k-1)z-uz(1+(2k+1)z) = 1-(2k-1+u)z-(2k+1)uz^2$. Then

$$F_{\mathbf{F}_X}(z,u) = \frac{g(z,u)}{P(z,u)}$$

and let $r(s)$ be the root of smallest modulus of $P(r(s), e^s) = 0$ in a small neighborhood of $s=0$. In particular $r(0) = \frac{1}{2k+1}$. According to [2, (3.1)], we obtain from [2, Theorem 1] that

$$\lim_{n\to\infty} \sup_x \Big| \sum_{k\leq \sigma_n x + \mu_n} p_n(k) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt \Big| = 0$$

with $\mu = \frac{r'(0)}{r(0)}$, $\mu_n = n\mu = n\frac{r'(0)}{r(0)}$ and $\sigma_n^2 = n\sigma^2 = n(\mu^2 - \frac{r''(0)}{r(0)})$.

Computing $r'(0)$ or easy combinatorial considerations gives $\mu_n = \frac{n}{k+1}$. The actual value of σ is here useless.

Now let $\epsilon < \frac{1}{k+1}$ and $\delta > 0$. Let x such that $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt < \delta$. Then there exists N such that for $n > N$, $\epsilon n < \sigma\sqrt{n}x + \frac{n}{k+1}$ since $\epsilon < \frac{1}{k+1}$. Therefore, for $n > N$,

$$\sum_{k<\epsilon n} p_n(k) \leq \sum_{k<\sigma_n x + \mu_n} p_n(k)$$

and there exists N_1 such that for $n > N_1$,

$$\sum_{k<\epsilon n} p_n(k) \leq 2\delta. \qquad \square$$

COROLLARY 3.2. *For $\#X = k$, $\#R = n$, $x_0 \in X$ and $0 < \epsilon < 1/k$ fixed, being (ϵ, x_0)-balanced is generic for $\Gamma = \langle X | R \rangle$.*

Proof of corollary. We choose n relations at random; by Lemma 3.1, every $r \in R$ is generically (ϵ, x_0)-balanced, but the conjunction of finitely many generic properties is also generic. □

4. SOME SUFFICIENT CONDITIONS FOR THE EXISTENCE OF FREE SUBGROUPS

We first begin by a very easy proposition.

PROPOSITION 4.1. *Let $\Gamma = \langle X \mid R \rangle$ be a finite presentation, which has a Dehn algorithm and such that for some $y \in X$ every subword u of every $r \in R^*$ with $|u| > |r|/2$ contains either y or y^{-1}, then $X - \{y\}$ generates a free subgroup in Γ.*

The proof of this proposition will follow from Lemma 4.2 below.

LEMMA 4.2. *For $\langle X | R \rangle$ a finite presentation of a group Γ and $y \in X$, the following are equivalent:*

- *$X - \{y\}$ freely generates a free subgroup of Γ;*
- *every non trivial element $\omega \in \mathbf{F}_X$, which represents the identity in Γ, contains either y or y^{-1}.*

Proof. 1) $\Rightarrow$ 2) : By contraposition, suppose that there exists a non trivial reduced element $\omega \in \mathbf{F}_{X-\{y\}}$ such that $\overline{\omega} = e$ (where $\overline{\omega}$ is the canonical projection of ω in Γ), then $X - \{y\}$ does not freely generate a free subgroup in Γ.

2) $\Rightarrow$ 1) : Let $\omega_1, \omega_2 \in \mathbf{F}_{X-\{y\}}$ be two reduced elements such that $\overline{\omega}_1 = \overline{\omega}_2 \in \Gamma$. Then $\overline{\omega_1\omega_2^{-1}} = e \in \Gamma$. So $\omega_1\omega_2^{-1}$ is an element of $\mathbf{F}_{X-\{y\}}$ which represents the identity in Γ. By hypothesis, this implies $\omega_1 = \omega_2$ in $\mathbf{F}_X$. Hence $X - \{y\}$ freely generates a free subgroup in Γ. □

Proof of Proposition 4.1. By Lemma 4.2, it is sufficient to show that every non trivial reduced word on $\mathbf{F}_X$ which represents the identity in Γ contains either y or y^{-1}. By assumption, $\Gamma = \langle X | R \rangle$ satisfies a Dehn algorithm, so such a word contains at least one half of a relator r in R which contains at least one occurrence of y or y^{-1}. □

The interest of this proposition appears when we replace "having a Dehn's algorithm" by "satisfying the small cancellation condition $C'(1/6)$", because $C'(1/6)$ and the fact that every subword u of any relation r with $|u| > |r|/2$ contains at least one y or y^{-1} are easy to check on a given presentation.

Unfortunately, as explained before, it is not known if the small cancellation hypothesis is generic, so we need other sufficient conditions to ensure that $X - \{y\}$ generates a free subgroup in Γ.

PROPOSITION 4.3. *Let $\Gamma = \langle X \mid R\rangle$ be a finite presentation with k generators and l relations, which is (ϵ, x_0)-balanced for some $0 < \epsilon < 1/k$ and some $x_0 \in X$, and which satisfies a θ-condition such that $\theta < \epsilon/(2-\epsilon)$. Then $X - \{x_0\}$ freely generates a free group in Γ.*

To prove the proposition we need the following lemma and the following notations. For a cell f_i of the diagram, we denote by $Int(f_i)$ (resp. by $Ext(f_i)$) the number of edges of f_i which are internal to the diagram (resp. which are on the border of the diagram). We denote also by $\#(f_i)$ the total number of edges of the cell f_i.

LEMMA 4.4. *Let $\Gamma = \langle X|R\rangle$ be a finite presentation of a group Γ which satisfies a θ-condition for some $0 < \theta < 1$, then for every reduced diagram, there exists a 2-cell f of Δ satisfying*

$$Int(f) \leq \frac{2\theta}{1+\theta}\#(f).$$

Proof. First we prove it for simple diagrams. Let $\epsilon = 2\theta/(1+\theta)$. Because the diagram is simple we have the following equalities:

I) $\sum_i Ext(f_i) = E(\Delta) = |\partial\Delta|$,

II) $\sum_i Int(f_i) = 2I(\Delta)$, because every internal edge belongs to two different cells.

So we get:

$$\#(\Delta) = \frac{1}{2}\sum_i Int(f_i) + \sum_i Ext(f_i) = \sum_i \#(f_i) - \frac{1}{2}\sum_i Int(f_i).$$

To obtain a contradiction, we suppose that every cell f_i of one diagram Δ is such that $(1/\epsilon)Int(f_i) > \#(f_i)$. Then we have

$$\frac{1}{\epsilon}\sum_i Int(f_i) > \sum_i \#(f_i) = \#(\Delta) + \frac{1}{2}\sum_i Int(f_i),$$

whence $\frac{2-\epsilon}{2\epsilon}\sum_i Int(f_i) > \#(\Delta)$ or $\frac{2-\epsilon}{\epsilon}I(\Delta) > \#(\Delta)$. Since $\epsilon = 2\theta/(1+\theta)$, we obtain $I(\Delta) > \theta\#(\Delta)$, which contradicts the θ-condition.

In fact, if the reduced diagram Δ is not simple, it is a union of simple diagrams linked by bridges. So each of its parts, which is a simple diagram, defines another reduced diagram (relative to another word), so the inequality holds for every part of Δ which is a simple diagram. We conclude by saying that increasing the number of external edges does not affect the inequality. □

Proof of 4.3. By Lemma 4.2, it is sufficient to prove that the (ϵ, x_0)-balanced and θ-conditions imply that every non trivial reduced word in $\mathbf{F}_X$ which vanishes in Γ contains at least one $x_0^{\pm 1}$.

Let us choose such a word ω and Δ a reduced diagram of ω. By Lemma 4.4, there exists a cell f with border equal to one $r \in R^*$, such that

$$Int(f) \leq \frac{2\theta}{1+\theta}\#(f) = \frac{2\theta}{1+\theta}|r| < \epsilon|r| \leq n_{x_0}(r),$$

because $\theta < \epsilon/(2-\epsilon)$. As there are more occurences of x_0 or x_0^{-1} than the number of internal edges, it means that some occurrences of x_0 or x_0^{-1} will be external edges, i.e. will be in the border of Δ which is ω. □

We are now able to prove the main theorem.

Proof of theorem 1.1. By Proposition 4.3, for a finite presentation $\langle X \,|\, R\rangle$, we know that being (ϵ, x_0)-balanced and satisfying a θ-condition is sufficient to ensure that $X - \{x_0\}$ freely generates a free subgroup in Γ. But by Corollary 3.2 and [13, Theorem 2], these two conditions are generic and so is the conjunction of these two conditions. □

5. Spectral estimates for adjacency operators on Cayley graphs

The existence of a free subgroup generated by $X - \{x_0\}$ gives an upper bound for the spectral value of the adjacency operator on the Cayley graph of $\Gamma = \langle X \,|\, R\rangle$ associated with the symmetric generating system $S = X \cup X^{-1}$.

We briefly recall some definitions and notations. The Cayley graph $G(\Gamma, X)$ of Γ associated with S has its set of vertices in bijection with Γ and two vertices g_1 and g_2 are linked by an edge if and only $g_1^{-1}g_2 \in S$. A graph is completely determined by its adjacency operator and, in the case of Cayley graphs, the adjacency operator h_S can be expressed in terms of the right regular representation ρ acting on $l^2(\Gamma)$ as

$$h_S = \frac{1}{\#S}\sum_{s\in S}\rho(s)\,.$$

The spectral properties of h_S capture some information about the pair $(\Gamma, S = X \cup X^{-1})$. For example, Kesten proved

THEOREM (Kesten [10], [9]). *Let Γ be a finitely generated group, let X be a finite generating system and set $S = X \cup X^{-1}$.*

a) The following are equivalent:

i) $\|h_S\| = 1$;

ii) Γ is amenable.

b) Assume that $\#X \geq 2$; then $\frac{\sqrt{2(\#X)-1}}{\#X} \leq \|h_S\|$. Equality holds if and only if Γ is isomorphic to the free group $\mathbf{F}_X$ generated by X.

This enables us to give an easy proposition which was pointed out to us by Pierre de la Harpe.

PROPOSITION 5.1. *Let $\Gamma = \langle X \,|\, R\rangle$ be a finite presentation of a group Γ with $\#X \geq 2$. If $X \cap X^{-1} = \varnothing$ and if there exists $x_0 \in X$ such that $X - \{x_0\}$ generates a free subgroup in Γ then*

$$\frac{\sqrt{2(\#X)-1}}{\#X} \leq \|h_S\| \leq \frac{\sqrt{2(\#X)-3}+1}{\#X}\,.$$

Proof. The first inequality is just Kesten's. To prove the second one, set $X' = X - \{x_0\}$, $S' = X' \cup (X')^{-1}$. Then we can write

$$(\#S)h_S = \rho(x_0) + \rho(x_0)^{-1} + \sum_{s\in S'}\rho(s)\,.$$

As X' freely generates a free group, by Kesten's result, we obtain that

$$\Big\|\sum_{s\in S'}\rho(s)\Big\| = 2\sqrt{2(\#X')-1} = 2\sqrt{2(\#X)-3}\,.$$

So

$$\|(\#S)h_S\| \leq 2 + \Big\|\sum_{s\in S'}\rho(s)\Big\| = 2 + 2\sqrt{2(\#X)-3}\,.$$

And as $X \cap X^{-1} = \varnothing$, $\#S = 2\#X$ and we get

$$\|h_S\| \leq \frac{1+\sqrt{2(\#X)-3}}{\#X}\,.$$

□

The last proposition and Theorem 1.1 permit us to give generic upper bounds for $\|h_S\|$. Note that this upper bound is non trivial only for $\#X \geq 3$.

COROLLARY 5.2. *For a presentation* $\Gamma = \langle X \,|\, R\rangle$ *with* $\#X = k \geq 3$ *and* $\#R = m$ *fixed, the inequalities* $\frac{\sqrt{2(\#X)-1}}{\#X} \leq \|h_S\| \leq \frac{\sqrt{2(\#X)-3}+1}{\#X}$ *are generically true.*

Proof. Let $x, y \in X$. Since $k \geq 3$, there exists $x_0 \in X$ distinct from x and y. By Theorem 1.1, the subgroup generated by $X - \{x_0\}$ is generically free; in particular $xy \neq e$ in Γ. This shows that, generically, $X \cap X^{-1} = \varnothing$. The corollary follows then by combining Theorem 1.1 with Proposition 5.1. □

It was proved by Grigorchuk [7, Theorem 7.1] that for any fixed $\epsilon > 0$, any group satisfying the small cancellation hypothesis $C'(1/6)$ and such that the length of every relation is sufficiently large satisfies

$$\frac{\sqrt{2(\#X)-1}}{\#X} < \|h_S\| < \frac{\sqrt{2(\#X)-1}}{\#X} + \epsilon\,.$$

This corresponds to the intuitive idea that when the relations are long and do not cancel too much, the Cayley graph looks like a tree in some ball of large radius.

Champetier (in [4]) generalised this theorem, by replacing the small cancellation $C'(1/6)$, by a weaker condition defined by:

DEFINITION (Champetier). A finite presentation $\langle x_1, \ldots, x_k \mid r_1, \ldots, r_m\rangle$ satisfies the $A(C)$ condition for $C > 0$, if for every word ω in $\mathbf{F}_X$ representing the identity in $\langle X \,|\, R\rangle$, there exists a diagram Δ representing ω such that, if there are l_i 2-cells contained in Δ having the relation r_i as border, then

$$\sum_{i=1}^{m} l_i |r_i| \leq C|\omega|\,.$$

With that definition, the precise statement of Champetier's theorem is:

THEOREM (Champetier). *Let C be a positive constant. For every $\epsilon > 0$, there exists an integer n_0 such that for every presentation $\Gamma = \langle X \mid R \rangle$, with $\#R = m$, satisfying $A(C)$ and $n_0 \leq \inf\{|r| \mid r \in R\}$, the following inequalities hold:*

$$\frac{\sqrt{2(\#X) - 1}}{\#X} \leq \|h_S\| \leq \left(1 + \frac{\epsilon}{2}\right) \frac{\sqrt{2(\#X) - 1}}{\#X}.$$

Assume the presentation satisfies a θ-condition: then $I(\Delta) < \theta\#(\Delta) = \theta(I(\Delta) + |\omega|)$, for any reduced diagram associated with ω. As

$$\sum_{i=1}^{m} l_i |r_i| = \sum_{2\text{-cell } f \subset \Delta} \#(f) \leq 2I(\Delta) + E(\Delta),$$

it is easy to see that the θ-condition implies $A(\frac{2\theta}{1-\theta} + 1)$. So Champetier's theorem and the genericity of the θ-condition imply:

COROLLARY 5.3. *For every $\epsilon > 0$, every fixed $\#X = k$ and every fixed $\#R = m$, $\|h_S\|$ is generically close to $\frac{\sqrt{2(\#X)-1}}{\#X}$.*

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