3. The construction of a birational group

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The map $D \mapsto (\mathcal{L}(D), s_D)$ defines a one-to-one correspondence between the set of relative effective Cartier divisors on X/T and the isomorphism classes of pairs $(\mathcal{L}, s)$, where $\mathcal{L}$ is an invertible sheaf on X and s is a global section of $\mathcal{L}$ such that the map $s\colon \mathcal{O}_X \to \mathcal{L}$ induced by the section s is injective and $\mathcal{L}/s\mathcal{O}_X$ is $\mathcal{O}_T$-flat.

The proof of the following lemma is straightforward and is left to the reader:

LEMMA 2.2.

(a) *If D_1 and D_2 are relative effective Cartier divisors on X/T, then so is $D_1 + D_2$.*

(b) *Let D_1 and D_2 be two relative effective Cartier divisors on X/T and let $\mathcal{I}(D_1)$ and $\mathcal{I}(D_2)$ be their ideal sheaves. If $\mathcal{I}(D_1) \subset \mathcal{I}(D_2)$, then $D_1 - D_2$ is also a relative effective Cartier divisor on X/T.*

(c) *Let $T' \to T$ be a base extension and let $X' = X \times_T T'$. If D is a relative effective Cartier divisor on X/T, then its pull-back to a closed subscheme D' of X' is a relative effective Cartier divisor on X'/T'.*

LEMMA 2.3. *Assume $q\colon X \to T$ is flat. Let $\mathcal{I}$ be a coherent sheaf of ideals of $\mathcal{O}_X$ and let D be the closed subscheme of X defined by $\mathcal{I}$. If for every point $x \in D$, the ideal $\mathcal{I}_x$ of $\mathcal{O}_{X,x}$ is generated by one element g_x whose image in $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{T,q(x)}} k(q(x))$ is not a zero divisor, then D is a relative effective Cartier divisor.*

Proof. It suffices to show that g_x is not a zero divisor in $\mathcal{O}_{X,x}$ and that $\mathcal{O}_{X,x}/(g_x)$ is flat over $\mathcal{O}_{T,q(x)}$. This follows from [EGA] §0.10.2.4 by taking $A = \mathcal{O}_{T,q(x)}$, $B = \mathcal{O}_{X,x}$, $M = N = \mathcal{O}_{X,x}$, and $u\colon M \to N$ to be the homomorphism $g_x\colon \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}$ defined by the multiplication by g_x.

3. THE CONSTRUCTION OF A BIRATIONAL GROUP

Let X be a nonsingular irreducible projective curve over an algebraically closed field k. A *modulus* $\mathfrak{m}$ supported on a finite subset S of X is a divisor of the form $\mathfrak{m} = \sum_{P \in S} n_P P$ with each $n_P > 0$. For any rational function f on X, we write $f \equiv 0 \mod \mathfrak{m}$ if $v_P(f) \geq n_P$ for every $P \in S$, where v_P is the valuation defined by P. Two divisors D_1 and D_2 on X prime to S are called $\mathfrak{m}$-*equivalent* if there exists a rational function f satisfying $f - 1 \equiv 0 \mod \mathfrak{m}$ such that $D_1 - D_2 = (f)$. If this holds, we write $D_1 \sim_{\mathfrak{m}} D_2$. Define a ringed space $(X_{\mathfrak{m}}, \mathcal{O}_{X_{\mathfrak{m}}})$ as follows: The underlying set of $X_{\mathfrak{m}}$ is $(X - S) \cup \{Q\}$. Define

$$\mathcal{O}_{X_{\mathfrak{m}},Q} = k + \{f \mid f \equiv 0 \mod \mathfrak{m}\}$$

and for every $x \in X - S$, define $\mathcal{O}_{X_{\mathfrak{m}},x} = \mathcal{O}_{X,x}$. One can show that when $\deg(\mathfrak{m}) \geq 2$, the ringed space $X_{\mathfrak{m}}$ is a singular curve with a unique singular point Q and its normalization is X. (It is easy to see that when $\deg(\mathfrak{m}) < 2$, the ringed space $X_{\mathfrak{m}}$ is identified with X itself.) For a divisor D of X prime to S, we put

$$L_{\mathfrak{m}}(D) = H^0(X_{\mathfrak{m}}, \mathcal{L}_{\mathfrak{m}}), \quad I_{\mathfrak{m}}(D) = H^1(X_{\mathfrak{m}}, \mathcal{L}_{\mathfrak{m}}),$$

where $\mathcal{L}_{\mathfrak{m}}$ is the invertible sheaf on $X_{\mathfrak{m}}$ corresponding to D. Denote the dimensions of $L_{\mathfrak{m}}(D)$ and $I_{\mathfrak{m}}(D)$ by $l_{\mathfrak{m}}(D)$ and $i_{\mathfrak{m}}(D)$, respectively. The Riemann-Roch theorem states that

$$l_{\mathfrak{m}}(D) - i_{\mathfrak{m}}(D) = \deg(D) + 1 - \pi .$$

In this formula, π is the sum $\pi = g + \delta$, where g is the genus of X and $\delta = \deg(\mathfrak{m}) - 1$. All these results are proved in [S], Chapter IV.

For convenience, a closed point on a scheme is just called a point.

Let T be a connected k-scheme. Consider the Cartesian square

$$\begin{tikzcd} X_{\mathfrak{m}} \times T \arrow[r] \arrow[d, "q"'] & X_{\mathfrak{m}} \arrow[d] \\ T \arrow[r] & \operatorname{spec}(k) . \end{tikzcd}$$

Since $X_{\mathfrak{m}}$ is proper and flat over $\operatorname{spec}(k)$, the morphism q is also proper and flat. Let D be a relative effective Cartier divisor on $(X_{\mathfrak{m}} \times T)/T$ supported on $(X_{\mathfrak{m}} - Q) \times T$ and let $\mathcal{L}$ be the invertible sheaf corresponding to D. Applying Theorem 1.1 (a) to the morphism q and the invertible sheaf $\mathcal{L}$, we conclude that $t \mapsto \chi(\mathcal{L}_t)$ is a constant function on T. By the Riemann-Roch theorem, we have $\chi(\mathcal{L}_t) = \deg D_t + 1 - \pi$. So $\deg(D_t)$ is also a constant. This constant is called the *degree* of D. Denote by $\operatorname{Div}^{(n)}(T)$ the set of all relative effective Cartier divisors of degree n on $(X_{\mathfrak{m}} \times T)/T$ supported on $(X_{\mathfrak{m}} - Q) \times T$.

Let $(X - S)^{(n)}$ be the n-th symmetric power of $X - S$, i.e., the quotient of $(X - S)^n$ by the action of the n-th symmetric group $\mathfrak{S}_n$, where $\mathfrak{S}_n$ acts on $(X - S)^n$ by permuting the factors. In the Appendix we show that there exists a relative effective Cartier divisor $\mathcal{D} \in \operatorname{Div}^{(n)}\big((X - S)^{(n)}\big)$, called the *universal relative effective Cartier divisor*, whose restriction to the fiber of the projection $X_{\mathfrak{m}} \times (X - S)^{(n)} \to (X - S)^{(n)}$ at $P_1 + \cdots + P_n \in (X - S)^{(n)}$ is the divisor $P_1 + \cdots + P_n$ of $X_{\mathfrak{m}}$. Moreover, we have

PROPOSITION 3.1. *The functor* $T \mapsto \mathrm{Div}^{(n)}(T)$ *from the category of* k-*schemes to the category of sets is represented by the symmetric power* $(X-S)^{(n)}$. *More precisely, for any relative effective Cartier divisor* D *of degree* n *on* $(X_{\mathfrak{m}} \times T)/T$ *supported on* $(X_{\mathfrak{m}} - Q) \times T$, *there exists a unique morphism* $f\colon T \to (X-S)^{(n)}$ *such that the pull-back of* $\mathcal{D}$ *by* $\mathrm{id} \times f$ *is* D.

The proof of this proposition is given in the Appendix. The morphism $T \to (X-S)^{(n)}$ can be described as follows: For every $t \in T$, identifying the fiber of $q\colon X_{\mathfrak{m}} \times T \to T$ at t with $X_{\mathfrak{m}}$, we may regard the restriction D_t of D to the fiber at t as an effective divisor of degree n on $X_{\mathfrak{m}}$ supported on $X_{\mathfrak{m}} - Q$. But this kind of divisor can be thought of as a point in $(X-S)^{(n)}$. The morphism $T \to (X-S)^{(n)}$ is just $t \mapsto D_t$.

LEMMA 3.2. *Let* D *be a divisor of* X *prime to* S *such that* $i_{\mathfrak{m}}(D) \geq 1$. *Then there exists an open subset* U *of* $X-S$ *such that for every* $P \in U$, *we have* $i_{\mathfrak{m}}(D+P) = i_{\mathfrak{m}}(D) - 1$.

Proof. If $P \notin \mathrm{Supp}(D) \cup S$, then the dual vector space $I_{\mathfrak{m}}(D+P)^*$ of $I_{\mathfrak{m}}(D+P)$ is identified with the subspace of $I_{\mathfrak{m}}(D)^*$ formed by differential forms $\omega \in I_{\mathfrak{m}}(D)^*$ vanishing at P. Let $\{\omega_1, \ldots, \omega_{i_{\mathfrak{m}}(D)}\}$ be a basis of $I_{\mathfrak{m}}(D)^*$. We can then take U to be the complement of

$$\mathrm{Supp}(D) \cup S \cup \{P \mid \omega_i(P) = 0 \text{ for } i = 1, \ldots, i_{\mathfrak{m}}(D)\}.$$

LEMMA 3.3. *Let* D_0 *be a divisor of* X *prime to* S *of degree* 0. *Then the set*

$$V_{D_0} = \{D \in (X-S)^{(\pi)} \mid l_{\mathfrak{m}}(D+D_0) = 1 \text{ and } l(D+D_0-\mathfrak{m}) = 0\}$$

is non-empty and open in $(X-S)^{(\pi)}$.

Proof. Consider the Cartesian square

$$\begin{array}{ccc} X_{\mathfrak{m}} \times (X-S)^{(\pi)} & \xrightarrow{p} & X_{\mathfrak{m}} \\ {\scriptstyle q}\downarrow & & \downarrow \\ (X-S)^{(\pi)} & \longrightarrow & \mathrm{spec}(k)\,. \end{array}$$

Applying Theorem 1.1 (b) to q and the invertible sheaf $\mathcal{L}$ on $X_{\mathfrak{m}} \times (X-S)^{(\pi)}$ corresponding to the divisor $\mathcal{D} + p^*(D_0)$, where $\mathcal{D}$ is the universal relative effective Cartier divisor, we conclude that the set

$$V_1 = \{t \in (X-S)^{(\pi)} \mid \dim H^0(X_{\mathfrak{m}}, \mathcal{L}_t) \leq 1\}$$

is open, that is,

$$V_1 = \{D \in (X-S)^{(\pi)} \mid l_{\mathfrak{m}}(D+D_0) \leq 1\}$$

is open. By the Riemann-Roch theorem we have, for any $D \in (X-S)^{(\pi)}$,

$$l_{\mathfrak{m}}(D+D_0) \geq \deg(D+D_0) + 1 - \pi = 1.$$

So we must have

$$V_1 = \{D \in (X-S)^{(\pi)} \mid l_{\mathfrak{m}}(D+D_0) = 1\}.$$

If $l_{\mathfrak{m}}(D_0) \neq 0$, then there exists a rational function f on X such that $(f)+D_0$ is an effective divisor on X prime to S. This effective divisor must be 0 since it is of degree 0. Hence $l_{\mathfrak{m}}(D_0) = l_{\mathfrak{m}}((f)+D_0) = l_{\mathfrak{m}}(0) = 1$. So in any case we have $l_{\mathfrak{m}}(D_0) \leq 1$. By the Riemann-Roch theorem, we have $i_{\mathfrak{m}}(D_0) \leq \pi$. Applying Lemma 3.2 repeatedly, we can find $P_1, \ldots, P_{i_{\mathfrak{m}}(D_0)}$ in $X-S$ so that $i_{\mathfrak{m}}(D_0 + P_1 + \cdots + P_{i_{\mathfrak{m}}(D_0)}) = 0$. Choose $P_{i_{\mathfrak{m}}(D_0)+1}, \ldots, P_\pi$ in $X-S$ arbitrarily. We have

$$i_{\mathfrak{m}}(D_0+P_1+\cdots+P_{i_{\mathfrak{m}}(D_0)}) \geq i_{\mathfrak{m}}(D_0+P_1+\cdots+P_{i_{\mathfrak{m}}(D_0)}+P_{i_{\mathfrak{m}}(D_0)+1}+\cdots+P_\pi).$$

(This can be seen by interpreting $i_{\mathfrak{m}}(D)$ as the dimension of the vector space of differential forms ω regular at Q satisfying $(\omega) \geq D$.) So we have $i_{\mathfrak{m}}(D_0 + P_1 + \cdots + P_\pi) = 0$. By the Riemann-Roch theorem, we have $l_{\mathfrak{m}}(D_0 + P_1 + \cdots + P_\pi) = 1$. Hence $P_1 + \cdots + P_\pi$ is in the set V_1 and V_1 is not empty.

Similarly by Theorem 1.1 (b) applied to the projection $q\colon X \times (X-S)^{(\pi)} \to (X-S)^{(\pi)}$ and the invertible sheaf on $X \times (X-S)^{(\pi)}$ corresponding to the divisor $\mathcal{D} + p^*(D_0 - \mathfrak{m})$, where $p\colon X \times (X-S)^{(\pi)} \to X$ is another projection, we see that the set

$$V_2 = \{D \in (X-S)^{(\pi)} \mid l(D+D_0-\mathfrak{m}) = 0\}$$

is open. Since $\deg(D_0 - \mathfrak{m}) < 0$, we have $l(D_0-\mathfrak{m}) = 0$. By the Riemann-Roch theorem, we have $i(D_0 - \mathfrak{m}) = \pi$. Applying Lemma 3.2 repeatedly (but taking $\mathfrak{m} = 0$), we can find $P_1, \ldots, P_\pi \in X-S$ such that $i(D_0-\mathfrak{m}+P_1+\cdots+P_\pi) = 0$. Then by the Riemann-Roch theorem we have $l(D_0 - \mathfrak{m} + P_1 + \cdots + P_\pi) = 0$. So $P_1 + \cdots + P_\pi$ is in V_2 and V_2 is not empty.

Since $(X-S)^{(\pi)}$ is irreducible, the set $V_{D_0} = V_1 \cap V_2$ is open and non-empty.

LEMMA 3.4. *Fix a point P_0 in S.*

(a) *The set*

$$U = \{(D_1, D_2) \in (X-S)^{(\pi)} \times (X-S)^{(\pi)} \mid l_{\mathfrak{m}}(D_1 + D_2 - \pi P_0) = 1, \quad l(D_1 + D_2 - \pi P_0 - \mathfrak{m}) = 0\}$$

is a non-empty open subset of $(X-S)^{(\pi)} \times (X-S)^{(\pi)}$.

(b) *The set*

$$V = \{(D_1, D_2) \in (X-S)^{(\pi)} \times (X-S)^{(\pi)} \mid l_{\mathfrak{m}}(D_2 - D_1 + \pi P_0) = 1, \quad l(D_2 - D_1 + \pi P_0 - \mathfrak{m}) = 0\}$$

is a non-empty open subset of $(X-S)^{(\pi)} \times (X-S)^{(\pi)}$.

Proof. (a) Let $p_1, p_2 \colon (X-S)^{(\pi)} \times (X-S)^{(\pi)} \to (X-S)^{(\pi)}$ be the projections and let E_i $(i = 1, 2)$ be the pull-backs by $\mathrm{id} \times p_i$ of the universal relative effective Cartier divisor $\mathcal{D}$ on $X_{\mathfrak{m}} \times (X-S)^{(\pi)}$. Put $E = E_1 + E_2$. This is a divisor on $X_{\mathfrak{m}} \times (X-S)^{(\pi)} \times (X-S)^{(\pi)}$.

Consider the Cartesian square

$$\begin{array}{ccc} X_{\mathfrak{m}} \times (X-S)^{(\pi)} \times (X-S)^{(\pi)} & \xrightarrow{p} & X_{\mathfrak{m}} \\ {\scriptstyle q}\downarrow & & \downarrow \\ (X-S)^{(\pi)} \times (X-S)^{(\pi)} & \longrightarrow & \mathrm{spec}(k)\,. \end{array}$$

By the Riemann-Roch theorem, for any $(D_1, D_2) \in (X-S)^{(\pi)} \times (X-S)^{(\pi)}$, we have

$$l_{\mathfrak{m}}(D_1 + D_2 - \pi P_0) \geq \deg(D_1 + D_2 - \pi P_0) + 1 - \pi = 1\,,$$

that is, for any $t \in (X-S)^{(\pi)} \times (X-S)^{(\pi)}$, we have $l_{\mathfrak{m}}(E_t - \pi P_0) \geq 1$. Applying Theorem 1.1 (b) to the projection q and the invertible sheaf corresponding to the divisor $E - p^*(P_0)$, we see that the set

$$U_1 = \{t \in (X-S)^{(\pi)} \times (X-S)^{(\pi)} \mid l_{\mathfrak{m}}(E_t - \pi P_0) = 1\}$$

is open. Similarly the set

$$U_2 = \{t \in (X-S)^{(\pi)} \times (X-S)^{(\pi)} \mid l(E_t - \pi P_0 - \mathfrak{m}) = 0\}$$

is also open. Hence the set $U = U_1 \cap U_2$ is open.

Applying Lemma 3.3 to $D_0 = 0$, we see that there exists a $D \in (X-S)^{(\pi)}$ such that $l_{\mathfrak{m}}(D) = 1$ and $l(D - \mathfrak{m}) = 0$. Then $(D, \pi P_0)$ is in U. So U is non-empty. This proves (a).

The proof of (b) is similar and is omitted.

DEFINITION 3.5. A *birational group* over k is a nonsingular variety V together with a rational map $m\colon V \times V \to V$, $(a,b) \mapsto ab$ such that

(a) $(ab)c = a(bc)$ when both sides are defined;

(b) the rational maps $\Phi\colon (a,b) \mapsto (a, ab)$ and $\Psi\colon (a,b) \mapsto (b, ab)$ on $V \times V$ are birational.

PROPOSITION 3.6. *There exists a unique rational map*

$$m\colon (X-S)^{(\pi)} \times (X-S)^{(\pi)} \to (X-S)^{(\pi)}$$

whose domain of definition contains the set U *in* 3.4 (a) *such that* $m(D_1, D_2)$ *is the unique effective divisor that is* $\mathfrak{m}$*-equivalent to* $D_1 + D_2 - \pi P_0$ *for any* $(D_1, D_2) \in U$. *Moreover* m *makes* $(X-S)^{(\pi)}$ *a birational group.*

Proof. Keep the notations in the proof of Lemma 3.4. Consider the Cartesian squares

$$\begin{array}{ccccccc} X_{\mathfrak{m}} = q^{-1}(t) & \longrightarrow & X_{\mathfrak{m}} \times U & \subset & X_{\mathfrak{m}} \times (X-S)^{(\pi)} \times (X-S)^{(\pi)} & \xrightarrow{p} & X_{\mathfrak{m}} \\ \downarrow & & \downarrow{\scriptstyle q} & & \downarrow & & \downarrow \\ \operatorname{spec}(k(t)) & \longrightarrow & U & \subset & (X-S)^{(\pi)} \times (X-S)^{(\pi)} & \longrightarrow & \operatorname{spec}(k). \end{array}$$

Let $\mathcal{L}$ be the restriction to $X_{\mathfrak{m}} \times U$ of the invertible sheaf corresponding to the divisor $E_1 + E_2 - p^*(\pi P_0)$. By Theorem 1.1 (c) and the choice of U, the sheaf $q_*\mathcal{L}$ is invertible. The canonical homomorphism $q^*q_*\mathcal{L} \to \mathcal{L}$ gives rise to $s\colon \mathcal{O}_{X_{\mathfrak{m}} \times U} \to \mathcal{L} \otimes (q^*q_*\mathcal{L})^{-1}$. We claim that the pair $(\mathcal{L} \otimes (q^*q_*\mathcal{L})^{-1}, s)$ defines a relative effective Cartier divisor on $(X_{\mathfrak{m}} \times U)/U$. According to Remark 2.1, it is enough to check that s is injective and $\operatorname{coker}(s)$ is $\mathcal{O}_U$-flat. Since $\mathcal{L} \otimes (q^*q_*\mathcal{L})^{-1}$ is invertible, it is enough to verify s_t is injective for all $t \in U$ by [EGA] §0.10.2.4, where s_t is the homomorphism obtained by restricting s to the fiber of q at t. It suffices to show that the restriction of the canonical homomorphism $q^*q_*\mathcal{L} \to \mathcal{L}$ to the fiber of q at t is injective. By Theorem 1.1 (c) we have $q_*\mathcal{L} \otimes_{\mathcal{O}_U} k(t) = H^0(X_{\mathfrak{m}}, \mathcal{L}_t)$. So the restriction of the canonical homomorphism to the fiber is $H^0(X_{\mathfrak{m}}, \mathcal{L}_t) \otimes_k \mathcal{O}_{X_{\mathfrak{m}}} \to \mathcal{L}_t$. Denote this map by s'_t; we need to show it is injective. But we have $\dim H^0(X_{\mathfrak{m}}, \mathcal{L}_t) = 1$ since $t \in U$. If we fix a nonzero element $g \in H^0(X_{\mathfrak{m}}, \mathcal{L}_t)$, then s'_t is identified with $\mathcal{O}_{X_{\mathfrak{m}}} \to \mathcal{L}_t$, $a \mapsto ag$. This last map is injective since $X_{\mathfrak{m}}$ is an integral scheme and g can be thought of as a rational function. So s_t is injective. Hence $(\mathcal{L} \otimes (q^*q_*\mathcal{L})^{-1}, s)$ defines a relative effective Cartier divisor. The restriction of this divisor to the fiber of q at t is the divisor on $X_{\mathfrak{m}}$ defined by the pair $(\mathcal{L}_t, g)$, which is supported on $X_{\mathfrak{m}} - Q$. So the divisor defined by

$(\mathcal{L} \otimes (q^* q_* \mathcal{L})^{-1}, s)$ is supported on $(X_{\mathfrak{m}} - Q) \times U$. By Proposition 3.1 there exists a unique morphism of varieties $m\colon U \to (X - S)^{(\pi)}$ such that the divisor defined by $(\mathcal{L} \otimes (q^* q_* \mathcal{L})^{-1}, s)$ is the pull-back by $\mathrm{id} \times m$ of the universal relative effective Cartier divisor $\mathcal{D}$ on $X_{\mathfrak{m}} \times (X - S)^{(\pi)}$. For any $(D_1, D_2) \in U$, we have $l_{\mathfrak{m}}(D_1 + D_2 - \pi P_0) = 1$ and $l(D_1 + D_2 - \pi P_0 - \mathfrak{m}) = 0$. So there is one and only one effective divisor $\mathfrak{m}$-equivalent to $D_1 + D_2 - \pi P_0$ and it is simply $m(D_1, D_2)$.

Similarly, using Lemma 3.4 (b) and Proposition 3.1, one can show that there exists a morphism $r\colon V \to (X - S)^{(\pi)}$ such that $r(D_1, D_2)$ is the unique effective divisor $\mathfrak{m}$-equivalent to $D_2 - D_1 + \pi P_0$ for any $(D_1, D_2) \in V$.

Let us verify that m defines a birational group on $(X - S)^{(\pi)}$. First we show

$$m(m(D_1, D_2), D_3) = m(D_1, m(D_2, D_3))$$

when (D_1, D_2), (D_2, D_3), $(m(D_1, D_2), D_3)$ and $(D_1, m(D_2, D_3))$ all belong to U. Indeed $m(m(D_1, D_2), D_3)$ is the unique effective divisor $\mathfrak{m}$-equivalent to $m(D_1, D_2) + D_3 - \pi P_0$, and $m(D_1, m(D_2, D_3))$ is the unique effective divisor $\mathfrak{m}$-equivalent to $D_1 + m(D_2, D_3) - \pi P_0$. But $m(D_1, D_2) + D_3 - \pi P_0$ and $D_1 + m(D_2, D_3) - \pi P_0$ are $\mathfrak{m}$-equivalent since both are $\mathfrak{m}$-equivalent to $D_1 + D_2 + D_3 - 2\pi P_0$. So we have $m(m(D_1, D_2), D_3) = m(D_1, m(D_2, D_3))$.

One can also verify $m(D_1, D_2) = m(D_2, D_1)$ when both (D_1, D_2) and (D_2, D_1) are in U, that is, the operation m is commutative.

Next we show that $\Theta\colon (D_1, D_2) \mapsto (D_1, r(D_1, D_2))$ is the birational inverse of $\Phi\colon (D_1, D_2) \mapsto (D_1, m(D_1, D_2))$ so that Φ is birational. Since the operation m is commutative, the rational map $\Psi\colon (D_1, D_2) \mapsto (D_2, m(D_1, D_2))$ is also birational. Therefore m makes $(X - S)^{(\pi)}$ a birational group.

First we verify $\Phi\Theta(D_1, D_2) = (D_1, D_2)$ whenever the left-hand side is defined. We have

$$\Phi\Theta(D_1, D_2) = \Phi(D_1, r(D_1, D_2)) = (D_1, m(D_1, r(D_1, D_2))).$$

Moreover $m(D_1, r(D_1, D_2))$ is the unique effective divisor $\mathfrak{m}$-equivalent to $D_1 + r(D_1, D_2) - \pi P_0$. But D_2 is also an effective divisor $\mathfrak{m}$-equivalent to $D_1 + r(D_1, D_2) - \pi P_0$ since we have

$$D_1 + r(D_1, D_2) - \pi P_0 \sim_{\mathfrak{m}} D_1 + (D_2 - D_1 + \pi P_0) - \pi P_0 = D_2.$$

Hence $m(D_1, r(D_1, D_2)) = D_2$ and $\Phi\Theta(D_1, D_2) = (D_1, D_2)$.

Similarly one can show that $\Theta\Phi(D_1, D_2) = (D_1, D_2)$ when the left-hand side is defined.

Note that Φ is a regular morphism defined on U and Θ is a regular morphism defined on V. Since

$$\Phi\,\Theta(D_1, D_2) = (D_1, D_2) \quad \text{and} \quad \Theta\,\Phi(D_1, D_2) = (D_1, D_2)$$

whenever the left-hand sides are defined, the maps Φ and Θ induce regular morphisms $\Phi\colon U \cap \Phi^{-1}(V) \to V \cap \Theta^{-1}(U)$ and $\Theta\colon V \cap \Theta^{-1}(U) \to U \cap \Phi^{-1}(V)$. To show that Φ and Θ are birational inverses to each other, it is enough to check that $U \cap \Phi^{-1}(V)$ and $V \cap \Theta^{-1}(U)$ are non-empty.

Note that $(D_1, D_2) \in U \cap \Phi^{-1}(V)$ if and only if $(D_1, D_2) \in U$ and

$$l_{\mathfrak{m}}(m(D_1, D_2) - D_1 + \pi P_0) = 1, \quad l(m(D_1, D_2) - D_1 + \pi P_0 - \mathfrak{m}) = 0\,.$$

Since $m(D_1, D_2) \sim_{\mathfrak{m}} D_1 + D_2 - \pi P_0$, the above equations are equivalent to

$$l_{\mathfrak{m}}(D_2) = 1, \quad l(D_2 - \mathfrak{m}) = 0\,.$$

Applying Lemma 3.3 to the divisor $D_0 = 0$, we conclude that the set

$$V_0 = \{D \in (X - S)^{(\pi)} \mid l_{\mathfrak{m}}(D) = 0, \quad l(D - \mathfrak{m}) = 0\}$$

is open and non-empty. Since $(X - S)^{(\pi)} \times (X - S)^{(\pi)}$ is irreducible, the set $U \cap ((X - S)^{(\pi)} \times V_0)$ is also open and non-empty. This set is exactly $U \cap \Phi^{-1}(V)$. So $U \cap \Phi^{-1}(V)$ is non-empty.

Similarly $V \cap \Theta^{-1}(U)$ is also non-empty. This completes the proof of the proposition.

4. From birational groups to algebraic groups

Let k be an algebraically closed field, let V be a connected nonsingular variety over k, and let $m\colon V \times V \to V$, $(a, b) \mapsto ab$ be a rational map satisfying $(ab)c = a(bc)$. Assume the rational maps $\Phi(a, b) = (a, ab)$ and $\Psi(a, b) = (b, ab)$ are birational. Then there exist open subsets X_Φ, Y_Φ, X_Ψ and Y_Ψ in $V \times V$ such that Φ induces an isomorphism $X_\Phi \cong Y_\Phi$ and Ψ induces an isomorphism $X_\Psi \cong Y_\Psi$. Put $Z = X_\Phi \cap Y_\Phi \cap X_\Psi \cap Y_\Psi$.

It is convenient to write the formulae for Φ^{-1} and Ψ^{-1} as $\Phi^{-1}(a, b) = (a, a^{-1}b)$ and $\Psi^{-1}(a, b) = (ba^{-1}, a)$.